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THE RIEMANN HOMOGENEOUS BOUNDARY VALUE PROBLEM IN THE CLASS OF ANALYTIC FUNCTIONS WITH THE GIVEN INDICATOR UNDER THE ENTIRE ORDER

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ABSTRACT

The Riemann homogeneous boundary value problem is investigated in the class of piecewise analytic functions. The necessary and sufficient conditions are obtained for the existence of the solution.

1. INTRODUCTION

In the complex plane the contour L is given. It consists from m going to the infinity rays $L_j = \{\arg z = \beta_j\}$, $0 = \beta_1 < \beta_2 < \ldots < \beta_m < 2\pi$. Set the function

$$h_j(\theta), \quad \theta \in [\beta_j, \beta_{j+1}], j = 1, \ldots, m, \beta_{m+1} = 2\pi \tag{1.1}$$

which is trigonometrically p-convex at each of the segments $[\beta_j, \beta_{j+1}]$. At points $\theta = \beta_j$ there exist finite unilateral derivatives. The problems for such functions are given in [1], [2].

Definition 1.1 1. p. 69.The positive continuously differentable in $[0, \infty)$ function $\rho(r)$, satisfying the conditions

$$\lim_{r\to\infty} \rho(r) = \rho \in (0, \infty), \lim_{r\to\infty} \rho'(r) r \ln r = 0$$

is called the proximate order.

Let consider the Riemann homogeneous boundary value problem

$$\Phi^{+}(t) = G(t)\Phi^{-}(t), t \in L, \tag{1.2}$$

the coefficient $G(t)$ obeys the conditions

$$G(t) = \exp\{2\pi i \varphi_j(t)|t|^{\rho(|t|)}\}, t \in L_j, \varphi_j(t) \in H,$$

$$\begin{aligned} \varphi_j(\infty) &= \lambda_j + i\nu_j, \quad j = 1, \dots, m, & (1.3)\\ \varphi(0) &= 0. & (1.4) \end{aligned}$$

We shall find the solution of the problem in the class A_h of piecewise analytic functions, continuously extended to the edges of the cut along the contour L and having in the closure of every angular domains $D_j = \{\beta_j < \arg z < \beta_{j+1}\}$ the given indicator $h_j(\varphi)$ under the proximate order $\rho(r)$. Let consider the case when ρ is the natural number. At $\rho \neq [\rho]$ the problem is studied in [3].

Let introduce the canonic function of the Riemann problem according to the formula

$$X(z) = \exp\{\frac{z^{\rho+1}}{2\pi i} \int\limits_L \frac{\ln G(\tau) d\tau}{\tau^{\rho+1}(\tau - z)}\} \equiv \prod_{j=1}^{m} X_j(z). \tag{1.5}$$

2. AUXILARY RESULTS

We will prove two important results.

LEMMA 2.1. *The canonic function is the solution of the uniform Riemann problem in the class of the piecewise analytic functions, limited in any sector* $D_j \cap \{|z| < R\}$.

Proof. The statement of the lemma is the consequence of the Suchotsky formula. We also use the bounded integral of Cauchy type outside the neighbourhood of the infinitely distant point [4, p. 38, 16]. □

LEMMA 2.2. *Let* $\rho = [\rho]$. *Then the canonic function* $X(z)$ *will have the proximate order* $\rho(r)$ *in every domain* D_j *only in the case when there exists the finite limit*

$$\lim_{r \to \infty} L^{-1}(r) \int\limits_0^r \left(\sum_{j=1}^{m} e^{-i\rho\beta_j} \varphi_j(xe^{i\beta_j}) \right) x^{\rho(x)-\rho-1} dx = a + ib, \tag{2.1}$$

where $L(r) = r^{\rho(r)-\rho}$ *is a weakly changing function [2, p.48]. If the conclitious [5] are bulfilled; then in each of the closed regions* D_j *the canonic function has the totally regular growth (t.r.g.) with the order* $\rho(r)$ *and its indicator* $h_x(\theta)$ *satisfies the relations:*

a) for every $\epsilon > 0$ *there exists a uniform limit relative to* $\theta \in [\beta_j + \epsilon, \beta_{j+1} - \epsilon]$

$$h_x(\theta) = \lim_{r\to\infty} r^{-\rho(r)} \ln |X(r(re^{i\theta}))| = \sum_{j=1}^{m} (\theta - \beta_j - \pi sgn(\theta - \beta_j))$$

$$(\nu_j \cos\rho(\theta - \beta_j) + \lambda_j \sin\rho(\theta - \beta_j)) + a\cos\rho\theta - b\sin\rho\theta, \tag{2.2}$$

$$b) h_{x\pm}(0) = \overline{\lim_{r\to\infty}} r^{-\rho(r)} \ln |X^{\pm}(re^{i\beta_j})| = h_x(\beta_j \pm 0). \tag{2.3}$$

Proof. Let find first the asymptotic of the function

$$X_1(z) = \exp\{z^{\rho+1} \int_0^\infty \frac{\varphi_1(x) x^{\rho(x)} dx}{x^{\rho+1}(x-z)}\}.$$

Write it in the following form

$$\ln X_1(z) = -z^{\rho} \int_0^r \varphi_1(x) x^{\rho(x)-\rho-1} + z^{\rho}(\int_0^r \frac{(\varphi_1(x) - (\lambda_1 + i\nu_1))x^{\rho(x)} dx}{x^{\rho}(x-z)}$$

$$+z\int_r^\infty \frac{(\varphi_1(x) - (\lambda_1 + i\nu_1))x^{\rho(x)} dx}{x^{\rho}(x-z)}) + (\lambda_1 + i\nu_1) z^{\rho}(\int_0^r \frac{x^{\rho(x)-\rho} dx}{(x-z)}$$

$$+z\int_r^\infty \frac{x^{\rho(x)-\rho} dx}{(x-z)}) \equiv (I_1 + I_2 + I_3)(z).$$

Since $\varphi_1(x)$ satisfies the Holder condition then, according to [4, p. 78], we have

$$I_2(re^{i\theta}) = o(r^{\rho(r)}), \theta \in [0, 2\pi], r \to \infty.$$

For the function $I_3(z)$ at $\theta \in [\epsilon, 2\pi - \epsilon]$ the following relations exist [2, p.92,93]

$$I_3(re^{i\theta}) = -i(\lambda_1 + i\nu_1)(\theta - \pi) e^{i\rho\theta} r^{\rho(r)} + o(r^{\rho(r)}), r \to \infty.$$

Consequeutly, we have

$$\ln X_1(re^{i\theta}) = -(\lambda_1 + i\nu_1)(\theta - \pi)e^{i\rho\theta}r^{\rho(r)} + I_1(re^{i\theta}) + o(r^{\rho(r)}), r \to \infty.$$

Isolating here the real part, we obtain

$$\ln |X_1(re^{i\theta})| = (\theta - \pi)(\nu_1 \cos p\theta + \lambda_1 \sin \rho\theta)r^{\rho(r)} + ReI_1(re^{i\theta}) + o(r^{\rho(r)}).$$

Writing $I_1(z)$ in the form

$$I_1(re^{i\theta}) = -L^{-1}(r)(e^{i\rho\theta}\int\limits_0^r \varphi_1(x)x^{\rho(x)-\rho-1}dx)r^{\rho(r)},$$

anologous to [5] we come to the representation

$$\ln |X_1(re^{i\theta})| = r^{\rho(r)}\{\sum_{j=1}^{m}(\theta - \beta_j - \pi sgn(\theta - \beta_j))(\nu_1 \cos(\theta - \beta_j)$$

$$+\lambda_j \sin \rho(\theta - \beta_j)) + o(1)\} - r^{\rho(r)}Re\{L^{-1}(r)\int\limits_0^r \left(\sum_{j=1}^{m} e^{-i\rho\beta_j}\varphi_j(xe^{i\beta_j})\right)$$

$$x^{\rho(x)-\rho-1}dx\} \equiv (J_0 + J_1)(re^{i\theta}), \theta \in \cup_{j=1}^{m}[\beta_j + \epsilon, \beta_{j+1} - \epsilon] \equiv T. \tag{2.4}$$

Now we will prove the first statement of the lemma. Let assume that function $X(z)$ has t. r. g. under the order $\rho(r)$. Since the relation is defined

$$J_0(re^{i\theta}) = h_0(\theta)r^{\rho(r)} + o(r^{\rho(r)}), \theta \in T,$$

then it follows from (8) that there exists a weak limit [1, p. 182] when r tends to the infinity, we not assume the meanings of some sets of E with zero relative measure

$$\overset{*}{\lim_{r\to\infty}} J_1(re^{i\theta}) = -\overset{*}{\lim_{r\to\infty}} L^{-1}(r)Re\{e^{i\rho\theta}\int\limits_0^r u(x)dx\} = h_x(\theta) - h_0(\theta), \theta \in T,$$

where

$$u(x) = \left(\sum_{j=1}^{m} e^{-i\rho\beta_j}\varphi_j(xe^{i\beta_j})\right) x^{\rho(x)-\rho-1}.$$

If $\theta \in T$ then it follows from the linear independence of functions $\cos \rho\theta$ and $\sin \rho\theta$ that there exists a weak limit

$$\overset{*}{\lim_{r\to\infty}} L^{-1}(r) \int\limits_0^r u(x)dx = a + ib. \tag{2.5}$$

Analogous to [1, p. 204] it is proved that the weak limit in the equality (9) can be replaced by the ordinary limit and it is equal to the fulfilment of the condition (5).

If the condition (5) is fulfilled, then the canonic function will have the order $\varphi(\rho)$ in domains D_j. It follows from (8) that the statement a) of the lemma holds. It remains to prove the characteristic b) from which the continuity of the indicator $h_x(\theta)$ follows on the segments $[\beta_j, \beta_{j+1}], j = 1, \ldots, m$ and consequently in t.r.g. of the canonic function in closed domains $\overline{D_j}$ [6, p.47]. Using the Suchotsky formula we obtain

$$\ln |X_1^{\pm}(t)| = \pm i\pi\varphi_1(t)t^{\rho(t)} + I_1(t) + I_2(t) + I_3(t).$$

The relation $I_2(t) = 0(t^{\rho(t)}), t \to \infty$ is stated above, and the analogous condition for $I_3(t)$ is given by I.V.Ostrovsky [7]. Now, isolating the real part in the last equality, we have

$$\ln |X_1^{\pm}(t)| = \pm i\pi\nu_1 t^{\rho(t)} + J_1(t) + 0(t^{\rho(t)}), t \to \infty.$$

From this, taking into account the existence of the limit (5) and the continuity of indicators $h_{x_j}(\theta), j \neq 1$ at the point $\beta_1 = 0$, we obtain the statement b) lemma on the ray L_1. The proof of the equalities (7) on the other rays is analogous. □

Consequence 1. If there exists the finite limit (5),then it is necessary to fulfil the equality

$$\sum_{j=1}^{m} (\lambda_j + i\nu_j)e^{-i\rho\beta_j} = 0. \tag{2.6}$$

3. MAIN RESULTS

THEOREM 1. *Let $\rho = [\rho]$ and there exists the finite limit (5). Then in order to solve the Riemann homogeneous value problem (2), (3) in class A_h it is necessary and sufficient that the relations hold*

$$\begin{aligned} 2\pi\nu_j &= h(\beta_j - 0) - h(\beta_j + 0), 2\pi\rho\lambda_j \geq & (3.1)\\ & h'(\beta_j - 0) - h'(\beta_j + 0), \quad j = 1, \ldots, m. & (3.2)\end{aligned}$$

A general solution of the problem is given by the formula

$$\Phi(z) = X(z)F(z), \tag{3.3}$$

where $X(z)$ is the canonic function and $F(z)$ is the entire function, having under the proximate order $\rho(r)$ the indicator

$$h_F(\theta) = h(\theta) - h_x(\theta), \theta \in [0, 2\pi]. \tag{3.4}$$

Proof. Necessity. Let assume that the problem has solutions in class A_h. Then analogous to [6, p.20] the representation (12) is proved. As $X(z)$ has t.r.g. in every domain $\overline{D_j}$, then at $\theta \in [\beta_j, \beta_{j+1}], j = 1, \ldots, m$ the equality [2, p.208] $h_\Phi(\theta) = h_x(\theta) + h_F(\theta)$ holds, from which the formula (13) follows. As $F(z)$ is the entire function, then its indicator is 2π - trigonometrically periodical p-convex function. Thus, the conditions [2, p.76,77] are necessary fulfilled

$$h_F(\beta_j - 0) = h_F(\beta_j + 0), h'_F(\beta_j - 0) \leq h'_F(\beta_j + 0) \tag{3.5}$$

or

$$h(\beta_j - 0) - h(\beta_j + 0) = h_x(\beta_j - 0) - h_x(\beta_j + 0),$$

$$h'(\beta_j - 0) - h'(\beta_j + 0) \leq h'_x(\beta_j - 0) - h'x(\beta_j + 0).$$

Having calculated by formula (6) the values of the right-hand parts of these relations, we obtain the conditions (11).

Sufficiency. Let the conditions (11) are fulfilled when any solution of the problem $\Phi(z)$ is given by the formula (12). Then the equality (13) follows from the stated characteristics of the canonic functions , from which the relations (14) follow.

As $h(\theta)$ is trigonometrically p-convex at each of the segments $[\beta_j, \beta_{j+1}]$ according to the condition, and the indicator $h_x(\theta)$ is given by the formula (6), then the difference $h(\theta) - h_x(\theta) = h_F(\theta)$ is also trigonometrically convex on these segments. Thus it follows from (14) due to the characteristic feature of trigonometrically p-convex functions [2, p.79], that $h_F(\theta)$ is 2π - periodical and trigonometrically p-convex on R function.

Thus, in the considered case there exist the entire functions with the given indicator $h_F(\theta)$ [2, p. 124], which according to the formula (12) give the solution of the problem from class A_h. □

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