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Two Almost Homoclinic Solutions for a Class of Perturbed Hamiltonian Systems Without Coercive Conditions*

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Abstract. In this paper we consider the existence of almost homoclinic solutions for the following second order perturbed Hamiltonian systems

$$\ddot{u} - L(t)u + \nabla W(t,u) = f(t), \tag{PHS}$$

where $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ and $\nabla W(t,u)$ is the gradient of $W(t,u)$ at u, $f \in C(\mathbb{R}, \mathbb{R}^n)$ and belongs to $L^2(\mathbb{R}, \mathbb{R}^n)$. The novelty of this paper is that, assuming $L(t)$ is bounded in the sense that there are two constants $0 < \tau_1 < \tau_2 < \infty$ such that $\tau_1|u|^2 \le (L(t)u,u) \le \tau_2|u|^2$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}^n$, $W(t,u)$ satisfies Ambrosetti–Rabinowitz condition and some other reasonable hypotheses, $f(t)$ is sufficiently small in $L^2(\mathbb{R}, \mathbb{R}^n)$, we obtain some new criterion to guarantee that (PHS) has at least two nontrivial almost homoclinic solutions. Recent results in the literature are generalized and significantly improved.

Keywords: homoclinic solutions, critical point, variational methods, mountain pass theorem.

AMS Subject Classification: 34C37; 35A15; 37J45.

1 Introduction

The purpose of this work is to deal with the existence of almost homoclinic solutions for the following second order perturbed Hamiltonian systems

$$\ddot{u} - L(t)u + \nabla W(t,u) = f(t), \tag{PHS}$$

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where $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $\nabla W(t,u)$ is the gradient of $W(t,u)$ at u, $f \in C(\mathbb{R}, \mathbb{R}^n)$ and belongs to $L^2(\mathbb{R}, \mathbb{R}^n)$. As usual, we say that a solution $u(t)$ of (PHS) is almost homoclinic (to 0) if $u \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$. If $u(t) \not\equiv 0$, $u(t)$ is called a nontrivial almost homoclinic solution.

If $f(t) \equiv 0$, then (PHS) reduces to the following Hamiltonian systems

$$\ddot{u} - L(t)u + \nabla W(t,u) = 0. \tag{HS}$$

For this case, an almost homoclinic solution is a homoclinic solution. It is well known that the existence of homoclinic solutions for Hamiltonian systems and their importance in the study of the behavior of dynamical systems have been recognized from Poincaré [19]. They may be "organizing centers" for the dynamics in their neighborhood. From their existence one may, under certain conditions, infer the existence of chaos nearby or the bifurcation behavior of periodic orbits. In the past thirty years, with the works of [17] and [21] variational methods and critical point theory have been successfully applied for the search of the existence and multiplicity of homoclinic solutions of (HS). Assuming that $L(t)$ and $W(t,u)$ are independent of t or periodic in t, many authors have studied the existence of homoclinic solutions of (HS), see for instance [2, 4, 5, 6, 9, 18, 21, 33] and the references therein.

If $L(t)$ and $W(t,u)$ are neither autonomous nor periodic in t, this problem is quite different from the periodic systems, because of the lack of compactness of the Sobolev embedding, such as [1, 4, 7, 13, 17, 22] and the references mentioned there. It is worth pointing out that to obtain the existence of homoclinic solutions of (HS), the so-called global Ambrosetti–Rabinowitz condition ((AR) condition) on $W(t,u)$ due to Ambrosetti–Rabinowitz (e.g., [3]) is assumed in the works mentioned above. Explicitly,

(W_1) there is a constant $\theta > 2$ such that, for every $t \in \mathbb{R}$ and $u \in \mathbb{R}^n \setminus \{0\}$,

$$0 < \theta W(t,u) \le \big(\nabla W(t,u), u\big),$$

which implies that $W(t,u)$ is of superquadratic growth as $|u| \to \infty$, where $(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the standard inner product in $\mathbb{R}^n$ and subsequently $|\cdot|$ is the induced norm. In addition, to verify (PS) condition for the corresponding functional of (HS), the following coercive assumption on $L(t)$ is often supposed:

(L) $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there is a continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) > 0$ for all $t \in \mathbb{R}$ and $(L(t)u, u) \ge \alpha(t)|u|^2$ and $\alpha(t) \to \infty$ as $|t| \to \infty$, which indicates that the smallest eigenvalue $l(t)$ of $L(t)$ is coercive, i.e.,

$$l(t) \to \infty \quad \text{as } |t| \to \infty, \tag{1.1}$$

where $l(t) = \inf_{|u|=1}(L(t)u, u)$. More recently, many authors discussed the existence of homoclinic solutions of (HS) under some superquadratic conditions on $W(t,u)$ which are weaker than (AR) condition, see for instance [16, 27] and the references listed therein. Furthermore, in mathematical physics, it is of frequent occurrence in (HS) that the global positive definiteness of $L(t)$ is not satisfied. In [6], for the first time, the author considered this case and obtained the existence and multiplicity of homoclinic solutions of (HS), in terms of one type of coercive condition on $L(t)$, which has been improved in recent papers [26, 29, 32] by introducing some coercive conditions different from (1.1) (these coercive conditions are used to obtain the corresponding compact embedding theorem), when $W(t,u)$ is superquadratic as $|u| \to \infty$. However, different types of coercive conditions on $L(t)$ mentioned above do not seem to be natural and are restrictive. For example, if $L(t) = \tau I_n$ (where $\tau > 0$ is a constant and I_n is the unit matrix of order n), then it is not covered by the above coercive conditions. In [13], the authors showed that the condition (1.1) can be removed if $L(t)$ and $W(t,u)$ are even in t, which has been improved in [16]. Besides, if $L \in C^1(\mathbb{R}, \mathbb{R}^{n^2})$, without assuming that $L(t)$ and $W(t,u)$ are even functions in t, the authors [14] obtained the existence of homoclinic solutions of (HS).

Compared with the literature available for $W(t,u)$ being superquadratic as $|u| \to \infty$, the study of the existence of homoclinic solutions of (HS) under the assumption that $W(t,u)$ is subquadratic at infinity is much more recent and the number of references is considerably smaller, see recent papers [6,23,34,35], where some types of coercive conditions on $L(t)$ are also utilized. In addition, the existence of homoclinic solutions for the case that $W(t,u)$ is asymptotically quadratic at infinity has also been investigated by many researchers in recent papers [8, 28, 36, 37].

In present paper, we are interested in the existence of almost homoclinic solutions for the perturbed Hamiltonian systems (PHS). Recently, many authors have focused their attention on (PHS) and showed that (PHS) possessed at least one nontrivial almost homoclinic solution. For example, see [10, 15, 25, 31] for the periodic systems and [11, 24, 30] for the nonperiodic systems. As far as we know, only the author in [12] proved that (PHS) possesses at least two almost homoclinic solutions, assuming that (L) holds, $W(t,u)$ satisfies (W_1) and some other reasonable hypothesis on $W(t,u)$ and $f(t)$, see its Theorem 1.1.

As pointed out above, the coercive condition (L) is very restrictive. In present paper we are mainly interested in the case that $L(t)$ is bounded in the sense that

(L)′ $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there are two constants $0 < \tau_1 < \tau_2 < \infty$ such that

$$\tau_1 |u|^2 \le \big(L(t)u, u\big) \le \tau_2 |u|^2 \quad \text{for all } (t,u) \in \mathbb{R} \times \mathbb{R}^n.$$

In this case we assume that the potential $W(t,u)$ satisfies (W_1) and the following condition:

(W_2) there exists some positive continuous function $a : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{|t| \to \infty} a(t) = 0 \tag{1.2}$$

such that

$$\big|\nabla W(t,u)\big| \le a(t) |u|^{\theta - 1} \quad \text{for all } (t,u) \in \mathbb{R} \times \mathbb{R}^n.$$

For the statement of our main result, we also need some estimation on L^2 norm of f. To this end, define

$$E = \Big\{ u \in H^1(\mathbb{R}, \mathbb{R}^n) : \int_{\mathbb{R}} \big[|\dot{u}(t)|^2 + \big(L(t)u(t), u(t)\big) \big] dt < \infty \Big\}.$$

Then the space E is a Hilbert space with the inner product

$$(u, v)_E = \int_{\mathbb{R}} \big[\big(\dot{u}(t), \dot{v}(t)\big) + \big(L(t)u(t), v(t)\big) \big] dt$$

and the corresponding norm is $\|u\|^2 = (u, u)_E$. Let $L^p(\mathbb{R}, \mathbb{R}^n)$ $(2 \le p < \infty)$ and $H^1(\mathbb{R}, \mathbb{R}^n)$ denote the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norms

$$\|u\|_p := \Big(\int_{\mathbb{R}} |u(t)|^p dt \Big)^{1/p} \quad \text{and} \quad \|u\|_{H^1} := \big(\|u\|_2^2 + \|\dot{u}\|_2^2 \big)^{1/2},$$

respectively. From (L)$'$, it is obvious that there exists $\beta > 0$ such that

$$\|u\|_2 \le \|u\|_{H^1} \le \beta \|u\|, \quad \forall u \in E. \tag{1.3}$$

Letting $\varrho = \sup\{W(t, u) : t \in \mathbb{R},\ |u| = 1\}$ and supposing that

(W$_3$) $\varrho < \frac{1}{2\beta^2}$ and $f : \mathbb{R} \to \mathbb{R}^n$ is a continuous square integrable function such that

$$\lim_{|t| \to \infty} f(t) = 0 \tag{1.4}$$

and

$$\|f\|_2 < \sqrt{2} \Big(\frac{1}{2\beta^2} - \varrho \Big), \tag{1.5}$$

then we are in the position to state our main result.

Theorem 1. *Under the assumptions of* (L)$'$ *and* (W$_1$)–(W$_3$), (PHS) *has at least two nontrivial almost homoclinic solutions.*

Remark 1. Note that in (L)$'$, we assume that $L(t)$ is bounded. Therefore, the smallest eigenvalue of $L(t)$ does not tend to ∞ as $|t| \to \infty$, i.e., $L(t)$ need not satisfy the various coercive conditions in the above mentioned papers. Thus, the recent results in [11, 12, 24, 30] are generalized and improved significantly. Meanwhile, compared to the case that $f(t) \equiv 0$, we do not require that $L(t)$ is even in t or $L \in C^1(\mathbb{R}, \mathbb{R}^{n^2})$. Thus, the recent results in [6, 13, 14, 16, 26, 29, 32] are generalized and improved significantly.

Remark 2. As mentioned above, the coercive conditions are used to establish some compact embedding theorems to guarantee that (PS) condition (or the other weak compactness conditions) holds, which is the essential step to obtain the existence of homoclinic solutions of (PHS) via Mountain Pass Theorem. In present paper, we assume that $L(t)$ is bounded and could not obtain some compact embedding theorem. Therefore, one difficulty is to adapt some new technique to overcome this difficulty and test that (PS) condition is verified, see Lemmas 3 and 4 below.

Moreover, we must point out that the author in [11] only investigated almost homoclinic solutions of (PHS) in the sense that $u(t) \to 0$ as $|t| \to \infty$. Therefore, in order to obtain the existence of almost homoclinic solutions of (PHS) as usual, another difficulty is to verify that, under the conditions of Theorem 1, $\dot{u}(t) \to 0$ as $|t| \to \infty$ as well, see Lemma 2 below.

The remaining part of this paper is organized as follows. Some preliminary results are presented in Section 2. In Section 3, we are devoted to accomplishing the proof of Theorem 1.

2 Preliminary Results

In order to prove Theorem 1 via the critical point theory, we firstly recall some properties of the space E on which the variational framework associated with (PHS) is defined. Denote by $L^\infty(\mathbb{R}, \mathbb{R}^n)$ the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|u\|_\infty := \operatorname{ess\,sup}\{|u(t)| : t \in \mathbb{R}\}.$$

In view of (1.3) and Proposition 2.2 in [11], we have

Proposition 1. *For every $u \in H^1(\mathbb{R}, \mathbb{R}^n)$,*

$$\|u\|_\infty \leq \frac{\sqrt{2}}{2}\|u\|_{H^1} \leq \frac{\sqrt{2}\beta}{2}\|u\|. \tag{2.1}$$

Remark 3. In fact, according to (1.3) and (2.1), the embedding $E \hookrightarrow L^p(\mathbb{R}, \mathbb{R}^n)$ is continuous, where $p \in [2, \infty]$. That is, for any $p \in [2, \infty]$ there is $C_p > 0$ such that

$$\|u\|_p \leq C_p\|u\|, \quad \forall u \in E. \tag{2.2}$$

Proposition 2. [10, Fact 2.1] *Under the assumption of* (W$_1$)*, we have*

(i) $W(t, u) \leq W(t, \frac{u}{|u|})|u|^\theta$ *for* $t \in \mathbb{R}$ *and* $0 < |u| \leq 1$*;*

(ii) $W(t, u) \geq W(t, \frac{u}{|u|})|u|^\theta$ *for* $t \in \mathbb{R}$ *and* $|u| \geq 1$.

Now we introduce some more notations and necessary definitions. Let $\mathcal{B}$ be a real Banach space, $I \in C^1(\mathcal{B}, \mathbb{R})$ means that I is a continuously Fréchet-differentiable functional defined on $\mathcal{B}$. Recall that $I \in C^1(\mathcal{B}, \mathbb{R})$ is said to satisfy (PS) condition if any sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{B}$, for which $\{I(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence in $\mathcal{B}$.

Moreover, let B_r be the open ball in $\mathcal{B}$ with the radius r and centered at 0 and ∂B_r denotes its boundary. Under the conditions of Theorem 1, we obtain the existence of the first almost homoclinic solution of (PHS) by using of the following well-known Mountain Pass Theorem, see [20].

Lemma 1. [20, Theorem 2.2] *Let $\mathcal{B}$ be a real Banach space and $I \in C^1(\mathcal{B}, \mathbb{R})$ satisfying* (PS) *condition. Suppose that $I(0) = 0$ and*

(A1) *there are constants* ρ, $\alpha > 0$ *such that* $I|_{\partial B_\rho} \geq \alpha$, *and*

(A2) *there is an* $e \in \mathcal{B} \setminus \overline{B}_\rho$ *such that* $I(e) \leq 0$.

Then I *possesses a critical value* $c \geq \alpha$. *Moreover* c *can be characterized as*

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I\big(g(s)\big),$$

where $\Gamma = \big\{g \in C\big([0,1],\mathcal{B}\big) : g(0) = 0,\ g(1) = e\big\}$.

As far as the second one is concerned, we obtain it by minimizing method, which is contained in a small ball centered at 0, see Step 4 in the proof of Theorem 1.

3 The Proof of Theorem 1

Now we are going to establish the corresponding variational framework to obtain almost homoclinic solutions of (PHS). To this end, define the functional $I : \mathcal{B} = E \to \mathbb{R}$ by

$$\begin{aligned} I(u) &= \int_{\mathbb{R}} \Big[\frac{1}{2}\big|\dot{u}(t)\big|^2 + \frac{1}{2}\big(L(t)u(t), u(t)\big) - W\big(t, u(t)\big) + \big(f(t), u(t)\big)\Big]dt \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \big[W\big(t, u(t)\big) - \big(f(t), u(t)\big)\big]dt. \end{aligned} \tag{3.1}$$

Under the conditions of Theorem 1, we have

$$I'(u)v = \int_{\mathbb{R}} \big[\big(\dot{u}(t), \dot{v}(t)\big) + \big(L(t)u(t), v(t)\big) - \big(\nabla W(t, u(t)), v(t)\big) + \big(f(t), v(t)\big)\big]dt \tag{3.2}$$

for all u, $v \in E$, which yields that

$$I'(u)u = \|u\|^2 - \int_{\mathbb{R}} \big[\big(\nabla W\big(t, u(t)\big), u(t)\big) - \big(f(t), u(t)\big)\big]dt. \tag{3.3}$$

Moreover, I is a continuously Fréchet-differentiable functional defined on E, i.e., $I \in C^1(E, \mathbb{R})$.

Lemma 2. *Any critical point* u *of* I *on* E *is an almost homoclinic solution of* (PHS) *such that* $u(t) \to 0$ *and* $\dot{u}(t) \to 0$ *as* $|t| \to \infty$.

Proof. It is well known that $E \subset C^0(\mathbb{R}, \mathbb{R}^n)$ (the space of continuous functions u on $\mathbb{R}$ such that $u(t) \to 0$ as $|t| \to \infty$). Now, if $u \in E$ is a critical point of I, we deduce from (3.2) that $L(t)u - \nabla W(t, u) + f(t)$ is the weak derivative of $\dot{u}$. Recall that $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$, $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $f \in C(\mathbb{R}, \mathbb{R}^n)$, we thus have u is indeed in $C^2(\mathbb{R}, \mathbb{R}^n)$. In what follows, we show that $\dot{u}(t) \to 0$ as $|t| \to \infty$ as well. We only consider the case that $\dot{u}(t) \to 0$ as $t \to +\infty$. The other case is similar. In fact, if we denote

$$x(t) = \dot{u}(t), \tag{3.4}$$

then, in view of (L)$'$, (W_2) and (W_3), it deduces that

$$\dot{x}(t) = \ddot{u}(t) = L(t)u - \nabla W(t,u) + f(t) \to 0$$

as $t \to +\infty$. That is, for any $\epsilon > 0$, there exists $R_1 > 0$ such that

$$\big|\dot{x}(t)\big| < \epsilon, \quad \forall t \geq R_1. \tag{3.5}$$

On the other hand, since $\dot{u}(t) \in L^2(\mathbb{R},\mathbb{R})$, then $x(t) \in L^2(\mathbb{R},\mathbb{R})$. Therefore, there is $R_2 > 0$ such that

$$\int_{R_2}^{+\infty} |x|^2 dt < \epsilon^2.$$

For any $t > R = \max\{R_1, R_2\}$, there exists $\tilde{R} > R$ such that $t \in [\tilde{R}, \tilde{R}+1]$ and

$$\int_{\tilde{R}}^{\tilde{R}+1} |x|^2 dt < \epsilon^2, \tag{3.6}$$

which implies that there is at least $t_0 \in [\tilde{R}, \tilde{R}+1]$ such that

$$\big|x(t_0)\big| < \epsilon.$$

Combining this with (3.5), we obtain that

$$\big|x(t) - x(t_0)\big| = \left|\int_{t_0}^{t} \dot{x}(s)ds\right| \leq \int_{\tilde{R}}^{\tilde{R}+1} \big|\dot{x}(s)\big|ds < \epsilon, \quad t \in [\tilde{R}, \tilde{R}+1].$$

Consequently, we have

$$\big|x(t)\big| < 2\epsilon, \quad \forall t \geq R.$$

In view of (3.4), it is obvious that

$$\dot{u}(t) \to 0 \quad \text{as } t \to +\infty. \quad \square$$

Lemma 3. *Under the conditions of Theorem 1, Φ' is compact, i.e., $\Phi'(u_n) \to \Phi'(u)$ if $u_n \rightharpoonup u$ in E, where $\Phi : E \to \mathbb{R}$ is defined by*

$$\Phi(u) = \int_{\mathbb{R}} W(t,u)dt.$$

Proof. Assume that $u_n \rightharpoonup u$ in E, then there is some constant $M > 0$ such that

$$\|u_n\| \leq M \quad \text{and} \quad \|u\| \leq M$$

for $n \in \mathbb{N}$. In addition, from (W_2), for any $\epsilon > 0$ there exists $R > 0$ such that

$$\big|\nabla W(t,u)\big| \leq \epsilon |u|^{\theta-1} \quad \text{and} \quad \big|\nabla W(t,u_n)\big| \leq \epsilon |u_n|^{\theta-1} \tag{3.7}$$

for $|t| > R$.

Consequently, for n large enough, we have

$$\begin{aligned}
\big|\big(\Phi'(u_n)-\Phi'(u)\big)v\big| &\le \int_{\mathbb{R}}\big|\nabla W(t,u_n)-\nabla W(t,u)\big||v|dt\\
&\le \int_{-R}^{R}\big|\nabla W(t,u_n)-\nabla W(t,u)\big||v|dt\\
&\quad+\int_{|t|>R}\big|\nabla W(t,u_n)\big||v|dt+\int_{|t|>R}\big|\nabla W(t,u)\big||v|dt\\
&\le \epsilon\|v\|_\infty+\epsilon\int_{|t|>R}|u_n|^{\theta-1}|v|dt+\epsilon\int_{|t|>R}|u|^{\theta-1}|v|dt\\
&\le \epsilon C_\infty\|v\|+\epsilon\int_{|t|>R}\Big(\frac{\theta-1}{\theta}|u_n|^\theta+\frac{1}{\theta}|v|^\theta\Big)dt\\
&\quad+\epsilon\int_{|t|>R}\Big(\frac{\theta-1}{\theta}|u|^\theta+\frac{1}{\theta}|v|^\theta\Big)dt\\
&\le \epsilon C_\infty\|v\|+\epsilon\frac{\theta-1}{\theta}\int_{|t|>R}\big(|u_n|^\theta+|u|^\theta\big)dt+\epsilon\frac{2}{\theta}\int_{|t|>R}|v|^\theta dt. \qquad (3.8)
\end{aligned}$$

Here we apply Young inequality:

$$ab\le\frac{a^p}{p}+\frac{b^q}{q},\quad a,b>0,\ p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1.$$

Consequently, we obtain that

$$\begin{aligned}
\big\|\Phi'(u_n)-\Phi'(u)\big\| &= \sup_{\|v\|=1}\left|\int_{\mathbb{R}}\big(\nabla W(t,u_n)-\nabla W(t,u),v\big)dt\right|\\
&\le \epsilon C_\infty+2\epsilon(C_\theta M)^\theta\frac{\theta-1}{\theta}+\epsilon C_\theta^\theta\frac{2}{\theta},
\end{aligned}$$

which yields that $\Phi'(u_n)\to\Phi'(u)$ as $u_n\rightharpoonup u$, that is, Φ' is compact. □

Lemma 4. *Under the conditions of Theorem* 1*, I satisfies* (PS) *condition.*

Proof. Assume that $\{u_k\}_{k\in\mathbb{N}}\subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k)\to 0$ as $k\to\infty$. Then there exists a constant $C>0$ such that

$$\big|I(u_k)\big|\le C \quad\text{and}\quad \big\|I'(u_k)\big\|_{E^*}\le C \qquad (3.9)$$

for every $k\in\mathbb{N}$, where E^* is the dual space of E.

Firstly, we show that $\{u_n\}_{k\in\mathbb{N}}$ is bounded. In fact, in view of (W_1), (3.1), (3.3) and (3.9), we obtain that

$$\begin{aligned}
C+\frac{C}{\theta}\|u_k\| &\ge I(u_k)-\frac{1}{\theta}I'(u_k)u_k\\
&=\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\|u_k\|^2-\int_{\mathbb{R}}\Big[W\big(t,u_k(t)\big)-\frac{1}{\theta}\big(\nabla W\big(t,u_k(t)\big),u_k(t)\big)\Big]dt\\
&\quad+\Big(1-\frac{1}{\theta}\Big)\int_{\mathbb{R}}\big(f(t),u_k(t)\big)dt
\end{aligned}$$

$$\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_k\|^2-\left(1-\frac{1}{\theta}\right)C_2\|f\|_2\|u_k\|.$$

Since $\theta > 2$, the boundedness of $\{u_k\}_{k\in\mathbb{N}}$ follows directly. Then the sequence $\{u_k\}_{k\in\mathbb{N}}$ has a subsequence, again denoted by $\{u_k\}_{k\in\mathbb{N}}$, and there exists $u \in E$ such that $u_k \rightharpoonup u$ weakly in E, which yields that

$$\big(I'(u_k)-I'(u)\big)(u_k-u)\to 0 \quad \text{as } k\to\infty. \tag{3.10}$$

Moreover, according to Lemma 3, we have

$$\Phi'(u_k)\to\Phi'(u) \tag{3.11}$$

as $k\to\infty$. Consequently, combining (3.10), (3.11) with the following equality

$$\big(I'(u_k)-I'(u)\big)(u_k-u)=\|u_k-u\|^2-\big(\Phi'(u_k)-\Phi'(u)\big)\big(u_k(t)-u(t)\big),$$

we obtain that $\|u_k-u\|\to 0$ as $k\to\infty$ and prove that (PS) condition holds. □

Now we are in the position to give the proof of Theorem 1. We divide its proof into four steps.

Proof. **Step 1** It is clear that $I(0)=0$ and $I\in C^1(E,\mathbb{R})$ satisfies (PS) condition by Lemma 4.

Step 2 We now show that there exist constants $\rho>0$ and $\alpha>0$ such that I satisfies condition (A1) of Lemma 1. Let $\rho=\sqrt{2}/\beta$, where β is defined in (1.3). Assume that $u\in E$ with $\|u\|\leq\rho$, then, by (2.1), it deduces that $\|u\|_\infty\leq 1$.

In consequence, combining this with (i) of Proposition 2, we obtain that

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2-\int_{\mathbb{R}} W\left(t,\frac{u(t)}{|u(t)|}\right)\big|u(t)\big|^\theta dt-\int_\infty \big(f(t),u(t)\big)dt \\ &\geq \frac{1}{2}\|u\|^2-\varrho\int_{\mathbb{R}}\big|u(t)\big|^2dt-\beta\|f\|_2\|u\| \\ &= \left(\frac{1}{2}-\varrho\beta^2\right)\|u\|^2-\beta\|f\|_2\|u\|, \quad \|u\|\leq\rho, \end{aligned} \tag{3.12}$$

where ϱ is defined in (W_3). The inequalities (1.5) and (3.12) imply that

$$I|_{\partial B_\rho}\geq\frac{1}{\beta^2}-2\varrho-\sqrt{2}\|f\|_2=\alpha>0.$$

Step 3 It remains to prove that there exists an $e\in E$ such that $I(e)\leq 0$ with $\|e\|>\rho$, where ρ is defined in Step 2. Choose $\varphi\in E$ such that $|\varphi(t)|=1$ for all $t\in[0,1]$. In view of (3.1) and (ii) of Proposition 2, we have, for every $s\in[1,\infty)$,

$$I(s\varphi)=\frac{s^2}{2}\|\varphi\|^2-\int_{\mathbb{R}}W\big(t,s\varphi(t)\big)dt+s\int_{\mathbb{R}}\big(f(t),\varphi(t)\big)dt \tag{3.13}$$

$$\leq\frac{s^2}{2}\|\varphi\|^2-s^\theta\int_0^1 W\left(t,\frac{\varphi(t)}{|\varphi(t)|}\right)\big|\varphi(t)\big|^\theta dt+sC_2\|f\|_2\|\varphi\| \tag{3.14}$$

$$\leq\frac{s^2}{2}\|\varphi\|^2-ms^\theta\int_0^1\big|\varphi(t)\big|^\theta dt+sC_2\|f\|_2\|\varphi\|, \tag{3.15}$$

where $m = \min\{W(t,u) : t \in [0,1],\ |u| = 1\}$. Since $\theta > 2$, (3.13) implies that $I(s\varphi) = I(e) < 0$ for some $s \gg 1$ with $\|s\varphi\| > \rho$, where ρ is defined in Step 2. By Lemma 1, I possesses a critical value $c_1 \geq \alpha > 0$ given by

$$c_1 = \inf_{g \in \Gamma} \max_{s \in [0,1]} I\big(g(s)\big),$$

where $\Gamma = \big\{g \in C\big([0,1], E\big) : g(0) = 0,\ g(1) = e\big\}$. Hence there is $0 \neq u_1 \in E$ such that $I(u_1) = c_1$ and $I'(u_1) = 0$. That is, the first nontrivial almost homoclinic solution of (PHS) exists.

Step 4 From (3.12), we see that I is bounded from below on $\overline{B_\rho(0)}$. Therefore, we can denote

$$c_2 = \inf_{\|u\| \leq \rho} I(u),$$

where ρ is defined in Step 1. Due to the fact that $I(0) = 0$, so $c_2 < c_1$. Then, there is a minimizing sequence $\{v_k\}_{k \in \mathbb{N}} \subset \overline{B_\rho(0)}$ such that

$$I(v_k) \to c_2 \quad \text{and} \quad I'(v_k) \to 0$$

as $k \to \infty$. That is, $\{v_k\}_{k \in \mathbb{N}}$ is a (PS) sequence. Furthermore, from Lemma 4, I satisfies (PS) condition. Therefore, c_2 is one nontrivial critical value of I (note that in our case $u(t) \equiv 0$ is not a solution of (PHS)). Consequently, there is $0 \neq u_2 \in E$ such that $I(u_2) = c_2$ and $I'(u_2) = 0$. That is, I has another nontrivial almost homoclinic solution. □

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