Mathematical Modelling and Analysis
Volume 7 Number 1, 2002, pages 1–10

STABILITY ANALYSIS OF SEIDEL TYPE MULTICOMPONENT ITERATIVE METHOD

V.N. ABRASHIN[1], R. ČIEGIS[2], V. PAKENIENĖ[2] and N.G. ZHADAEVA[1]

[1] *Institute of Mathematics of National Academy of Sciences of Belarus*
Surganov Str. 11, 220072, Minsk, Belarus
[2] *Vilnius Gediminas Technical University*
Saulėtekio al. 11, LT-2040, Vilnius, Lithuania
E-mail: `abrashin@im.bas-net.by, rc@fm.vtu.lt`

Received January 19, 2002; revised February 14, 2002

ABSTRACT

This paper deals with the stability analysis of multicomponent iterative methods for solving elliptic problems. They are based on a general splitting method, which decomposes a multidimensional parabolic problem into a system of one dimensional implicit problems. Error estimates in the L_2 norm are proved for the first method. For the stability analysis of Seidel type iterative method we use a spectral method. Two dimensional and three dimensional problems are investigated. Finally, we present results of numerical experiments. Our goal is to investigate the dependence of convergence rates of multicomponent iterative methods on the smoothness of the solution. Hence we solve a discrete problem, which approximates the $3D$ Poisson's problem. It is proved that the number of iterations depends weakly on the number of grid points if the exact solution and the initial approximation are smooth functions, both. The same problem is also solved by the Stability Correction iterative method. The obtained results indicate a similar behavior.

1. INTRODUCTION

In this paper we continue analysis of multicomponent iterative methods, which was started in [1; 4]. Let consider the system of linear equations

$$\sum_{\alpha=1}^{p} A_\alpha y = f, \tag{1.1}$$

which are usually obtained after approximation of elliptic PDE problems by finite-difference or finite-element schemes.

Iterative methods for solving systems of linear equations are investigated in many papers. A particular interest is given for problems approximating elliptic problems of PDE. Most efficient iterative methods are obtained by using the splitting method. Splitting methods for elliptic problems are reviewed in [5; 7; 8; 9]. In particular, [6; 7] present an alternating direction method, [5] describes factorization schemes. The convergence rate of such algorithms can be increased if optimal non-stationary iterative parameters are used for the definition of each iteration, see e.g. [7].

The content of this paper is organized as follows. In Section 2, we investigate the convergence of multicomponent iterative scheme in the L_2 norm. The results of [1; 4] are generalized for this norm. The spectral stability of two Seidel type multicomponent iterative schemes is investigated in Section 3. In this section we also investigate the stability of 3D Seidel scheme, the analysis is done using numerical experiments. Finally, in Section 4 we present results of numerical experiments. The convergence rate of multicomponent iterative scheme is investigated for a problem with a smooth solution. It is proved that the convergence rate depends on the smoothness of the initial approximation of the solution.

2. MULTICOMPONENT ALTERNATING DIRECTION SCHEME

In this section we investigate the convergence of the *multicomponent alternating direction* (MAD) scheme

$$\frac{\overset{s+1}{y}_\alpha - \overset{s}{\tilde{y}}}{\tau} + pA_\alpha\Big(\overset{s+1}{y}_\alpha - \overset{s}{y}_\alpha\Big) + \sum_{\beta=1}^{p} A_\beta \overset{s}{y}_\beta = f\,, \quad \alpha = 1,2,\dots,p\,, \tag{2.1}$$

$$\overset{s}{\tilde{y}} = \frac{1}{p}\sum_{\alpha=1}^{p} \overset{s}{y}_\alpha\,,$$

where $\overset{s}{y}_\alpha$ is the s-th iteration of y_α. This iterative method was proposed in [2]. The following theorem was proved in [1], see also [4].

Theorem 2.1. *Iterative scheme* (2.1) *produces a sequence converging unconditionally to the solution of problem* (1.1) *and the convergence rate is estimated as*

$$Q_p(\overset{s+1}{y}) \le \frac{1}{q} Q_p(\overset{s}{y})\,, \quad q = \min\left(1 + m\,p\,\tau,\ 1 + \frac{1}{2M\,p\,\tau}\right), \tag{2.2}$$

where we use notation:

$$Q_p(\overset{s}{y}) = \| \overset{s}{r} \|^2 + \frac{1}{p^2\,\tau^2} \| \overset{s}{v} \|_3^2 , \quad \overset{s}{r} = \sum_{\alpha=1}^{p} A_\alpha \overset{s}{y}_\alpha - f ,$$

$$\| \overset{s}{v} \|_3^2 = \sum_{\alpha,\beta=1,\, \alpha>\beta}^{p} \| \overset{s}{v}^{(\alpha,\beta)} \|^2 , \quad \overset{s}{v}^{(\alpha,\beta)} = \overset{s}{y}_\alpha - \overset{s}{y}_\beta ,$$

and m and M are the spectral estimates of the operator A:

$$m = \min_{1\le\alpha\le p} m_\alpha, \quad M = \max_{1\le\alpha\le p} M_\alpha .$$

The convergence of (2.1) is proved in a very special norm $Q_p(y)$, hence it is important to estimate the convergence in the L_2 norm. The main result of this section is given in the following theorem.

Theorem 2.2. *If operators A_α commute, then the following error estimate in the L_2 norm*

$$\|\overset{s}{\tilde{y}} - y\| \le \left(\frac{1}{m} + p\tau\right) \frac{1}{q^{s/2}} \sqrt{Q_p(\overset{0}{y})} . \tag{2.3}$$

is valid.

Proof. Let denote $\overset{s}{\rho} = \overset{s}{\tilde{y}} - y$ the error function. Using the equality

$$\overset{s}{y}_\alpha = \overset{s}{\tilde{y}} + \frac{1}{p} \sum_{\alpha=1}^{p} \overset{s}{v}^{(\alpha,\beta)}$$

and the definition of $\overset{s}{r}$, we obtain the equality

$$A\, \overset{s}{\rho} = -\sum_{\alpha=1}^{p} A_\alpha \Big(\frac{1}{p} \sum_{\beta=1}^{p} \overset{s}{v}^{(\alpha,\beta)}\Big) + \overset{s}{r} \ .$$

Hence it follows from the equation given above that

$$\overset{s}{\rho} = -\sum_{\alpha=1}^{p} B_\alpha \Big(\frac{1}{p} \sum_{\beta=1}^{p} \overset{s}{v}^{(\alpha,\beta)}\Big) + A^{-1}\, \overset{s}{r} ,$$

here $B_\alpha = A^{-1}A_\alpha = \Big(E + \sum\limits_{\beta=1,\,\beta\neq\alpha}^{p} A_\alpha^{-1}A_\beta\Big)^{-1}$. If operators $A\alpha, \alpha = 1,\dots,p$ commute, then $\|B_\alpha\| < 1$ and we obtain the estimates

$$\|\overset{s}{\rho}\| \le \|A^{-1}\|\,\|\overset{s}{r}\| + \frac{1}{p}\sum_{\alpha=1}^{p}\sum_{\beta=1}^{p}\|\overset{s}{v}{}^{(\alpha,\beta)}\| \le \frac{1}{m}\|\overset{s}{r}\| + \sqrt{2}\,\|\overset{s}{v}\|_3\,. \qquad (2.4)$$

It follows from (2.2) that

$$\|\overset{s}{r}\| \le \left(\left(\frac{1}{q}\right)^{s} Q_p(\overset{0}{y})\right)^{1/2}, \quad \|\overset{s}{v}\|_3 \le p\tau\left(\left(\frac{1}{q}\right)^{s} Q_p(\overset{0}{y})\right)^{1/2}. \qquad (2.5)$$

We now combine inequalities (2.5) with (2.4) to obtain the required estimate of the error. ■

3. SEIDEL–TYPE ITERATIVE SCHEME

In this section we investigate the convergence rate of Seidel-type iterative scheme:

$$\frac{\overset{s+1}{y}_\alpha - \overset{s}{y}{}^*_\alpha}{\tau} + \sum_{\beta=1}^{\alpha} A_\beta\,\overset{s+1}{y}_\beta + \sum_{\beta=\alpha+1}^{p} A_\beta\,\overset{s}{y}_\beta = f\,, \quad \alpha = 1,2,\dots,p\,, \qquad (3.1)$$

$$\overset{s}{y}{}^*_1 = \overset{s}{y}_1, \quad \overset{s}{y}{}^*_\alpha = 0.5\left(\overset{s}{y}_\alpha + \overset{s}{y}_{\alpha-1}\right).$$

We also consider a modified Seidel iterative scheme (3.1), when $\overset{s}{y}{}^*_1$ is computed by a symmetrical formula:

$$\overset{s}{y}{}^*_1 = 0.5\left(\overset{s}{y}_1 + \overset{s}{y}_p\right). \qquad (3.2)$$

We note that a modification of Seidel multicomponent iterative scheme was proposed in [3]. The stability analysis of this scheme proved that it is only conditionally stable. A quasi-optimal iterative parameter is obtained in [3] for $2D$ problem.

3.1. Spectral stability analysis of 2D scheme

To apply the discrete von Neumann stability criteria to problem (3.1), we write the global error as a series:

$$\overset{s}{e}_\alpha = \sum_{j=1}^{N-1}\sum_{k=1}^{N-1} \overset{s}{d}_{\alpha,jk}\,\sin(j\pi x_1)\sin(k\pi x_2)\,, \quad \alpha = 1,2\,,$$

Substituting this expansion into (3.1), we obtain the equation for coefficients

$$\overset{s+1}{\mathbf{d}}_{jk} = Q_2 \overset{s}{\mathbf{d}}_{jk} \;. \tag{3.3}$$

where $\mathbf{d}_{jk}$ is the column vector of spectral coefficients and Q_2 is the stability matrix of scheme (3.1)

$$Q_2 = \begin{pmatrix} \dfrac{1}{1+\tau\lambda_j} & \dfrac{-\tau\lambda_k}{1+\tau\lambda_j} \\ & \\ \dfrac{0.5(1-\tau\lambda_j)}{(1+\tau\lambda_j)(1+\tau\lambda_k)} & \dfrac{0.5(1+\tau\lambda_j)+\tau^2\lambda_j\lambda_k}{(1+\tau\lambda_j)(1+\tau\lambda_k)} \end{pmatrix}.$$

Now we consider the necessary conditions for the stability of scheme (3.1). The eigenvalues of the amplification matrix Q_2 satisfy the quadratic equation

$$q^2 - \left(1 + \frac{0.5(1-\tau\lambda_j)}{(1+\tau\lambda_j)(1+\tau\lambda_k)}\right) q + \frac{0.5}{1+\tau\lambda_j} = 0\,.$$

Theorem 3.1. *All eigenvalues of stability matrix Q_2 satisfy inequalities*

$$|q_{jk}| < 1, \quad 1 \le j,k \le N-1$$

unconditionally for any values of parameters τ and h.

Proof. Application of the Hurwitz criterion gives that $|q_{jk}| \le 1$ is satisfied if and only if

$$\frac{0.5}{1+\tau\lambda_j} < 1\,, \quad \left|1 + \frac{0.5(1-\tau\lambda_j)}{(1+\tau\lambda_j)(1+\tau\lambda_k)}\right| < 1 + \frac{0.5}{1+\tau\lambda_j}.$$

Simple computations prove that both inequalities are satisfied unconditionally. The theorem is proved. ■

Now we will investigate the stability of the modified Seidel iterative scheme (3.1)–(3.2). In this case, coefficients $\overset{s}{\mathbf{d}}_{jk}$ satisfy the following equation

$$\overset{s+1}{\mathbf{d}}_{jk} = \begin{pmatrix} \dfrac{0.5}{1+\tau\lambda_j} & \dfrac{0.5-\tau\lambda_k}{1+\tau\lambda_j} \\ & \\ \dfrac{0.5(1-\tau\lambda_j)}{(1+\tau\lambda_j)(1+\tau\lambda_k)} & \dfrac{0.5(1+\tau\lambda_j)+\tau^2\lambda_j\lambda_k}{(1+\tau\lambda_j)(1+\tau\lambda_k)} \end{pmatrix} \overset{s}{\mathbf{d}}_{jk} \;.$$

The eigenvalues of this amplification matrix satisfy the quadratic equation

$$q^2 - \left(1 - \frac{0.5\tau(\lambda_j + \lambda_k)}{(1+\tau\lambda_j)(1+\tau\lambda_k)}\right) q + \frac{0.5\tau(\lambda_j + \lambda_k)}{(1+\tau\lambda_j)^2(1+\tau\lambda_k)} = 0\,. \qquad (3.4)$$

Theorem 3.2. *All eigenvalues of stability matrix of modified Seidel-type iterative scheme satisfy inequalities*

$$|q_{jk}| < 1, \quad 1 \le j,k \le N-1$$

unconditionally for any values of parameters τ *and* h*.*

The proof of the theorem is obtained by applying the Hurwitz criterion for (3.4).

3.2. Spectral stability analysis of 3D iterative scheme

Let consider the model problem

$$\sum_{\alpha=1}^{3} A_\alpha y = (\lambda_j + \lambda_k + \lambda_l)\sin(j\pi x_1)\sin(k\pi x_2)\sin(l\pi x_3)\,, \qquad (3.5)$$

which has the exact solution

$$y = \sin(j\pi x_1)\sin(k\pi x_2)\sin(l\pi x_3)\,.$$

The solution of 3D scheme (3.1) - (3.2) can be represented as

$$\overset{s}{y}_\alpha = \overset{s}{d}_\alpha \sin(j\pi x_1)\sin(k\pi x_2)\sin(l\pi x_3)\,, \quad \alpha = 1,2,3\,,$$

where $d_\alpha,\ \alpha = 1,2,3$, are computed explicitly

$$\overset{s+1}{d}_\alpha = \frac{1}{1+\tau\lambda_\alpha}\left(\frac{\overset{s}{d}_\alpha + \overset{s}{d}_{\alpha-1}}{2} - \sum_{\beta=1}^{\alpha-1}\tau\lambda_\beta \overset{s+1}{d}_\beta - \sum_{\beta=\alpha+1}^{3}\tau\lambda_\beta \overset{s}{d}_\beta + \sum_{\beta=1}^{3}\tau\lambda_\beta\right).$$

Here we take $\overset{s}{d}_0 = \overset{s}{d}_p$. We estimate the error of the sth iteration $\overset{s}{d}_\alpha$ by the following formula

$$\overset{s}{e} = \max_{1\le\alpha\le3}\left|\,\overset{s}{d}_\alpha\,\right|.$$

Our goal is to investigate numerically the dependence of the convergence rate of the iterative scheme (3.1)–(3.2) on m and M, i.e. spectral estimates of the matrix A. Numerical experiments proved that such modification of Seidel iterative scheme is more efficient than the initial version of Seidel scheme (3.1).

Table 1.
The optimal value of τ as a function of m and M.

m	M	τ_0	$S(\tau_0)$
10	4000	0.00209	148
10	16000	0.00105	294
10	64000	0.00052	551
10	16000	0.00105	294
40	16000	0.00052	149
90	16000	0.00034	102

In Tab. 1 we present optimal values of τ and numbers of iterations $S(\tau_0)$ for different spectral intervals. We used $\varepsilon = 10^{-4}$ in these experiments.

It follows from results presented in Tab. 1 that

$$\tau_0 = \frac{c}{\sqrt{mM}}\,, \quad S(\tau_0) = \mathcal{O}\left(\sqrt{\frac{M}{m}}\right).$$

4. NUMERICAL EXAMPLES

In this section we will present results of numerical experiments. We apply the MAD iterative scheme (2.1) to the following problem

$$\sum_{\alpha=1}^{3} A_\alpha y = f\,, \quad (x_{1i}, x_{2j}, x_{3k}) \in \omega_h\,, \tag{4.1}$$

$$A_\alpha y = -\frac{y(x_{\alpha,i+1}) - 2y(x_{\alpha,i}) + y(x_{\alpha,i-1})}{h^2}\,,$$

where y is a grid function, defined on the uniform grid ω_h with $(N+1) \times (N+1) \times (N+1)$ grid points, covering the cube $[0,1] \times [0,1] \times [0,1]$. The exact solution y satisfies the boundary condition $y = 0$.

Our goal is to investigate the dependence of the convergence rate on the smoothness of the initial approximation. First, we solve (4.1) with the exact solution given by

$$y(x_1, x_2, x_3) = \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3)e^{(x_1+x_2+x_3)^2}\,. \tag{4.2}$$

In Tab. 2 we present sample results obtained using MAD iterative method. The first column lists the parameter τ, the second, third and fourth columns show the numbers of iterations, required to reduce the error, measured in the L_2 norm, on the $(N+1) \times (N+1) \times (N+1)$ grid by a factor $1/\varepsilon$. Functions $y_\alpha \equiv 0$ are taken as initial approximations. The accuracy of iterations was taken $\varepsilon = 10^{-4}$.

Table 2.
Convergence analysis of MAD scheme for a smooth solution.

τ	$N = 10$	$N = 20$	$N = 40$	$N = 80$
0.025	61	113	167	205
0.010	34	47	68	83
0.005	63	62	62	63
0.0025	121	120	120	120

We see that convergence rate of MAD iterative scheme depends very slightly on the number of grid points. This fact can be explained by taking into account the fact that the solution of (4.1) and the initial approximation both are smooth functions, and the iterative method is unconditional stable. Hence only low frequency modes of spectral representation of y are important in the analysis, since the total energy of high frequency range modes is below the specified accuracy ε. A similar property is valid for the other unconditionally stable iterative schemes. For example, let consider the *stabilization correction* (SC) iterative method (see, e.g. [5]):

$$\frac{y^{s+1/3} - \overset{s}{y}}{\tau} + A_1 y^{s+1/3} + A_2 \overset{s}{y} + A_3 \overset{s}{y} = f\,,$$
$$\frac{y^{s+2/3} - y^{s+1/3}}{\tau} + A_2 \left(y^{s+2/3} - \overset{s}{y} \right) = 0\,,$$
$$\frac{y^{s+1} - y^{s+2/3}}{\tau} + A_3 \left(y^{s+1} - \overset{s}{y} \right) = 0\,.$$

In Tab. 3 we present results obtained using SC iterative method for the same problem (4.1)–(4.2).

Table 3.
Convergence analysis of SC iterative scheme for a smooth solution.

τ	$N = 10$	$N = 20$	$N = 40$	$N = 80$
0.025	56	113	144	151
0.010	34	35	36	37
0.005	63	63	63	63
0.0025	121	120	120	121

The observed convergence rate of MAD iterative method becomes close to one established in Theorem 2.1 when we take a non-smooth initial approximation of the solution. We now choose the exact solution $y \equiv 0$ and consider the case with the initial approximation $\overset{0}{y} \equiv 1$ on the interior points of the grid. The results are presented in Tab. 4.

Now the spectral representation of the initial global error includes all modes, but high frequency modes still are decreasing sufficiently fastly.

Table 4.
Convergence analysis of MAD iterative scheme for a non-smooth initial approximation.

τ	$N=10$	$N=20$	$N=40$	$N=80$
0.015	41	130	422	1705
0.010	36	87	282	910
0.005	66	66	142	456
0.0025	261	258	256	256

Is the last example we added to the initial approximation three high order modes

$$\begin{aligned}\overset{0}{y}_{ijk} = 1 &+ \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3) \\ &+ \sin((N-1)\pi x_1)\sin((N-1)\pi x_2)\sin((N-1)\pi x_3) \\ &+ \sin\left(\tfrac{N-1}{2}\pi x_1\right)\sin((N-1)\pi x_2)\sin((N-1)\pi x_3) \\ &+ \sin\left(\tfrac{N-1}{2}\pi x_1\right)\sin\left(\tfrac{N-1}{2}\pi x_2\right)\sin((N-1)\pi x_3)\,.\end{aligned}$$

In Tab. 5 we present results obtained with MAD iterative method.

Table 5.
Convergence analysis of MAD iterative scheme for
a perturbed non-smooth initial approximation.

τ	$N=10$	$N=20$	$N=40$
0.020	97	405	1628
0.010	50	204	816
0.005	66	104	409
0.0025	128	126	206

CONCLUSIONS

The results of numerical experiments prove that iterative methods converge faster for systems of linear equations which approximate elliptic boundary-value problems. Such a property is valid for iterative methods which are unconditionally stable. In this case the high order modes of the error do not influence the convergence of the iterative method, assuming that the total energy of these modes is below the specified error tolerance. It is well known that for smooth functions high order modes decrease very fastly.

REFERENCES

[1] V. Abrashin, R. Čiegis and V. Pakalnytė. Spectral stability analysis of one splitting iterative scheme. In: A. Samarskii R. Čiegis and M. Sapagovas(Eds.), *Proc. of the 3rd*

International Conference FDS2000, September 1-4, 2000, Palanga, Lithuania, Finite difference schemes: theory and applications, IMI, Vilnius, 1 – 10, 2000.

[2] V.N. Abrashin. On the stability of multicomponent additive direction method. *Differentsialnyje uravnenyja*, **35**, 212 – 224, 1999. (in Russian)

[3] T.G. Aleinikova and R. Čiegis. Investigation of new iterative methods for solving multidimensional elliptic problems. *Differentsialnyje uravnenyja*, **29**, 1124 – 1129, 1993. (in Russian)

[4] R. Čiegis and V. Pakalnytė. Spectral convergence analysis of multicomponent iterative methods. *Differential Equations*, **37**(7), 982 – 986, 2001.

[5] G.I. Marchuk. *Splitting methods.* Nauka, Moscow, 1988. (in Russian)

[6] D. Peaceman and H. Rachford. The numerical solution of parabolic and elliptic differential equations. *SIAM*, **3**(1), 1955.

[7] A.A. Samarskii and E.S. Nikolajev. *Methods for solving difference equations.* Nauka, Moscow, 1978. (in Russian)

[8] A.A. Samarskii and P.N. Vabishchevich. *Additive schemes for problems of mathematical physics.* Nauka, Moscow, 1999. (in Russian)

[9] J.W. Thomas. *Numerical Partial Differential Equations. Texts in Applied Mathematics*, volume 22. Springer, New York, Berlin, Heidelberg, 1995.

Zeidelio tipo daugiakomponentinio itercinio metodo stabilumo analizė

V.N. Abrašinas, R. Čiegis, V. Pakenienė, N.G. Žadajeva

Šiame straipsnyje tęsiama daugiakomponentinių iteracinių metodų stabilumo analizė, pradėta ankstesniuose autorių darbuose. Įrodytas vienos schemos sprendinio konvergavimas L_2 normoje. Spektriniu metodu ištirtas dviejų Zeidelio tipo iteracinių schemų stabilumas dvimačiu atveju, įrodyta, kad dvimačiu atveju abi schemos yra nesąlygiškai stabilios. Trimačio uždavinio spektrinio stabilumo analizė atlikta skaitiškai. Įrodyta, kad modifikuotoji schema pasižymi didesniu konvergavimo greičiu.

Paskutiniame skyriuje pateikti skaičiavimo eksperimento rezultatai. Buvo sprendžiamas trimatis Puasono uždavinys, aproksimuotas standartine baigtinių skirtumų schema. Ištirta iteracinių metodų konvergavimo greičio priklausomybė nuo sprendinio ir pradinio artinio glodumo. Parodyta, kad baigtinių skirtumų schemoms konvergavimo greitis gali silpnai priklausyti nuo diskrečiojo tinklo mazgų skaičiaus, jei pradinė paklaida yra glodi funkcija. Daugiakomponentiniai iteraciniai metodai palyginti su stabilizuojančios pataisos metodu, kuris irgi yra nesąlygiškai stabilus.