Mathematical Modelling and Analysis
Volume 8 Number 1, 2003, pages 13–24

EXPLICIT SOLUTION OF THE RIEMANN BOUNDARY VALUE PROBLEM FOR DOUBLE PERIODIC FUNCTIONS IN CASE OF COMPOUND CONTOUR

T.I. GATALSKAJA

Department of Mathematics and Mechanics Belarusian State University
F. Skaryny av.4, 220050 Minsk, Belarus
E-mail: gatalskaya@bsu.by

Received September 18, 2002

ABSTRACT

It is obtained the explicit solution to the Riemann boundary value problem for a compound contour on the torus. An example is presented which illustrates theoretical results.

Key words: Riemann boundary value problem, periodic functions, compound contour

1. STATEMENT OF A PROBLEM

Let $\mathbb{C}/\Gamma$ be a closed Riemann surface (torus), where Γ is biperiodic lattice

$$\Gamma = \{2m\omega + 2m'\omega' \big| \ m,\ m' \in \mathbb{Z}\},$$

2ω and $2\omega'$ are fixed basic periods, $\mathrm{Im}\dfrac{\omega'}{\omega} > 0$. All calculations will be carried out on the basic parallelogram of periods, supposing that the points t, τ satisfy one of the relations $\tau = t + 2\omega$ or $\tau = t + 2\omega'$ (see Fig. 1).

Let us denote

$$A = 0,\ B = 2\omega,\ C = 2\omega + 2\omega',\ D = 2\omega'.$$

Let $L = \bigcup_{j=1}^{\nu} L_j$, $L \subset \sqcap$ be a compound piece-wise smooth orientable contour (graph), where L_j are smooth arcs, homeomorphic to the interval $(0, 1)$ of the real line.

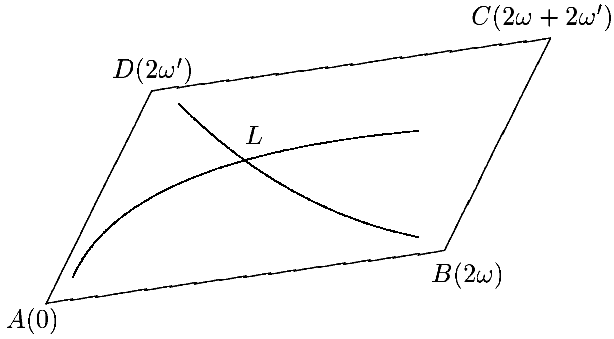

Figure 1. Basic parallelogram of periods: $\sqcap := \{2m\omega t + 2m'\omega' t' \mid 0 \leqslant t \leqslant 1\ ,\ 0 \leqslant t' \leqslant 1\}$.

Let the set $\Lambda \subset L$ consists of the end points of all L_j and of the common points of different arcs L_j. The points of Λ will be called knots of the contour L. Let $\mathcal{J}$ be a given divisor of order $m = \text{ord}\ \mathcal{J}$, consisting of points of set Λ, $\mathcal{D}$ be a given divisor of order $n = \text{ord}\ \mathcal{D}$, consisting of points of set $\sqcap \diagdown L$. Let $G(t),\ t \in L\diagdown\Lambda$ be a given nonvanishing piece-wise Hölder-continuous (H-continuous) function, continuously extendible up to end-points of all arcs L_j, $G(t_k) \neq 0$ for all end-points t_k. Let $g(t)\ ,\ t \in L\diagdown\Lambda$, be a given H-continuous function multiply to divisor $\mathcal{J}^{-1}$. We consider the following inhomogeneous Riemann boundary value problem:

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t),\ \ \mathcal{J}^{-1}\mathcal{D}^{-1}\big|'(\Phi),\ \ t \in L, \tag{1.1}$$

where $\big|'$ denotes pseudo-multiplicity (see [5]). Our goal is to find all double periodic functions Φ meromorphic on $\sqcap\diagdown L$, multiply to the divisor $\mathcal{D}^{-1}$, H-continuously extendible up to $L\diagdown\Lambda$, satisfying the boundary condition (1.1). We suppose also that Φ is pseudo-multiply to $\mathcal{J}^{-1}$ at knots of L.

The corresponding homogeneous problem has a form

$$\Phi^+(t) = G(t)\Phi^-(t),\ \ \mathcal{J}^{-1}\mathcal{D}^{-1}\big|(\Phi),\ \ t \in L. \tag{1.2}$$

Together with the problems (1.1) and (1.2) we will consider the so called associated problem (see [4]). In particular homogeneous boundary value problems associated with problem (1.2) is given by

$$\Psi^-(t) = G(t)\Psi^+(t)\ \ ,\ \ \mathcal{J}\mathcal{D}\ \Big|\big|(\Psi),\ \ t \in L, \tag{1.3}$$

where $\big|\big|$ denotes quasi-multiplicity (see [5])

2. INDEX OF THE PROBLEM

Let us determine an auxiliary function which will be used in the construction of the solution of both problems (1.1) and (1.2). We will find this function as a special solution of the homogeneous problem (1.2), which is analytic and nonvanishing on $\sqcap \setminus L$. In order to do this we first fix an arbitrary continuous branch of the function $\ln G_j(t)$ on each L_j. Let the arc L_j begins at a point t_k and terminates at a point t_l (probably $t_k = t_l$). Let us denote by $G_j(t_k - 0)$ and $G_j(t_k + 0)$ the limiting values of G_j at t_k from the right and from the left by orientation. We introduce the following notation

$$\varkappa_k = \frac{1}{2\pi}\Big(\sum_j{}' \arg G_j(t_k - 0) - \sum_j{}'' \arg G_j(t_k + 0)\Big),$$

where the sum $\sum''$ (respectively $\sum'$) contains all terms corresponding to arcs L_j which begin (terminate) at a point t_k. Numbers $\varkappa_k$ depend on the choice of branches of $\ln G_j(t)$. At the changes of branches the values of $\varkappa_k$ can be changed on integer numbers, but the following quantities

$$\sum_k \varkappa_k \ , \ \ \varkappa := \sum_k [\varkappa_k],$$

are invariant with respect to all possible changes of branches. Here $[x]$ denotes the largest integer smaller or equal to x.

The value $\varkappa$ is called index of coefficient $G(t)$ of problem (1.2). Let us consider a divisor $\mathcal{E}$: $\mathcal{E} = t_1^{[\varkappa_1]} t_2^{[\varkappa_2]} ... t_r^{[\varkappa_r]}$ of the order $\operatorname{ord} \mathcal{E} = \varkappa$.

We also introduce the following piece-wise analytic function $\chi(z) = e^{\Gamma(z)}$, where

$$\Gamma(z) = \frac{1}{2\pi i} \int_L \ln G(\tau) \zeta(\tau - z) d\tau,$$

and $\zeta(\cdot) = \zeta(\cdot \ , \omega \ , \omega')$ is the Weierstrass zeta-function (see [1]). We denote by $(\chi) = t_1^{\varkappa_1} t_2^{\varkappa_2} \dots t_r^{\varkappa_r}$ a quasi-divisor corresponding to the function $\chi(z)$ (see [5]). Let us determine the boundary values of the function $\chi(z)$ on the different sides of the lines L, AB and AD. Due to Sokhotsky-Plemelj formulas (see [4]) we have

$$\chi^+(t) = G(t)\chi^-(t), \ t \in L \setminus (\partial \sqcap \cup \Lambda).$$

Since we are searching a doubly periodic solution then the following relations

$$\chi(t) = C'\chi(t + 2\omega'), \ t \in AB \setminus L,$$

$$\chi(t) = C\chi(t + 2\omega), \ t \in AD \setminus L,$$

have to be valid with certain constants C and C'. Combining these relations we have

$$\chi^+(t) = \mp C'G(t)\chi^-(t)\ ,\ t \in (AB\setminus L) \sqcup (AB \cup L)\ ,$$

$$\chi^+(t) = \mp CG(t)\chi^-(t)\ ,\ t \in (AD\setminus L) \sqcup (AD \cup L)\ ,$$

$$C' = \exp\left\{\frac{\zeta(\omega')}{\pi i}\int_L \ln G(t)dt\right\}\ ,\ C = \exp\left\{\frac{\zeta(\omega)}{\pi i}\int_L \ln G(t)dt\right\}.$$

Here the sign "$-$" ("$+$") on the right-hand side is taken when the orientation of AB or AD coincides (differs) with the orientation of L at the point t.

3. THE SOLUTION OF THE RIEMANN BOUNDARY VALUE PROBLEM

Let us describe an algorithm for determination of the solution of homogeneous problem (1.2) (see [5]). In our case we put $h = 1$ and replace an analog of the Cauchy kernel by the expression $\zeta(\tau - t)d\tau$, where $\zeta(\cdot) = \zeta(\cdot\ ,\omega\ ,\omega')$. By [5] the solution of (1.2) is reduced to the solution of two successive problems – the Jacobi inversion problem and the construction of a doubly periodic function multiply to the given divisor.

3.1. The Jacobi inversion problem (see [5])

This problem consist in determination of points $q_1\ \ldots, q_h$ on a given Riemann surface $\Re$ of the genus h, and integer numbers $n_k,\ m_j$ satisfying the following relation

$$\sum_{j=1}^{h} w_k(q_j) = \frac{1}{2\pi i}\int_L \ln G(\tau)dw_k(\tau) + \sum_{j=1}^{h} B_{kj}m_j\ ,\ k = \overline{1,h}.$$

Here

$$w_k(q) = {\int_{\widetilde{q}}^{q}}' dw_k(t), \qquad \widetilde{q} \neq q_0$$

is an arbitrary fixed point on the surface, the symbol $\int'$ means that the contour of integration does not cross the canonical sections,

$$B_{kj} = \int_{b_k} dw_j(p), \qquad (k\ ,\ j = \overline{(1\ ,h)})$$

are B-periods of the complex-base $(dw_1, \dots, dw_h)$ of the Abel differentials of the first kind of the given surface $\Re$.

In our case ($h = 1$) the Jacobi inversion problem requires finding one point $\widetilde{z}$ and two integer numbers m, m'. Their existence is known a priori (see [5]).

3.2. Construction of the doubly periodic functions

Let $\widetilde{z}$ be a solution of the Jacoby inversion problem (a construction of $\widetilde{z}$ will be discussed later). Let us define a divisor $\mathcal{F} = (z_1) \cdot (\widetilde{z})^{-1}$, where $z_1 \in \sqcap^0 \setminus L$ is an arbitrary point, $\sqcap^0$ is an internal part of $\sqcap$.

In our notations the general solution of the homogeneous problem has a form

$$\Phi(z) = \widehat{\varphi}(z) \exp\Big\{ \frac{1}{2\pi i} \int\limits_L \ln G(\tau)\zeta(\tau - z)d\tau - \int\limits_{z_1}^{\widetilde{z}} \zeta(\tau - z)d\tau$$
$$- m \int\limits_{z_1}^{z_1+2\omega} \zeta(\tau - z)d\tau - m' \int\limits_{z_1}^{z_1+2\omega'} \zeta(\tau - z)d\tau \Big\}, \qquad (3.1)$$

where $\widehat{\varphi}$ is an arbitrary elliptic function, multiply to the given divisor $\mathcal{J}^{-1}\mathcal{D}^{-1}\mathcal{E}^{-1}\mathcal{F}^{-1}$. Here the branch of $\ln G(\tau)$ is fixed as above. Let us recall the following identities (see [1]):

$$\begin{aligned} \zeta(u - 2\omega) &= \zeta(u) - 2\zeta(\omega), \quad u \in [A, D], \\ \zeta(u - 2\omega') &= \zeta(u) - 2\zeta(\omega'), \quad u \in [A, B]. \end{aligned} \qquad (3.2)$$

Then the exponential term in (3.1) will be doubly periodic if and only if

$$z_1 + \frac{1}{2\pi i} \int\limits_L \ln G(\tau)d\tau = \widetilde{z} + 2m\omega + 2m'\omega'. \qquad (3.3)$$

This relation is a particular case of the Jacobi inversion problem. We determine now integer numbers m and m' and a point $\widetilde{z} \in [0, 2\omega) \times [0, 2\omega')$ from (3.3). Since ω and ω' are linear independent over the field $\mathbb{R}$, we can represent the left-hand side of (3.3) in the form

$$z_1 + \frac{1}{2\pi i} \int\limits_L \ln G(\tau)d\tau = 2\alpha\omega + 2\alpha'\omega' \ , \quad \text{where } \alpha,\ \alpha' \in \mathbb{R}.$$

Combining this relation with (3.3), we obtain:

$$m = [\alpha] \ , \ m' = [\alpha'] \ , \ \widetilde{z} = 2\{\alpha\}\omega + 2\{\alpha'\}\omega',$$

where [·] and {·} denote integer entire and fractional parts of a real number. Here $\widetilde{z}$ is the solution of Jacobi inversion problem. Hence the divisor $\mathcal{F} = (z_1)(\widetilde{z})^{-1}$ is completely determined.

We can simplify the left-hand side of (3.1). Taking into account that $\zeta(z) = \dfrac{d}{dz}\ln\sigma(z)$, where $\sigma(\cdot)$ is the Weierstrass sigma-function, we have:

$$\begin{aligned}
\Phi(z) &= \widehat{\varphi}(z)\exp\Big\{\frac{1}{2\pi i}\int\limits_L \ln G(\tau)\zeta(\tau-z)d\tau - \int\limits_{z_1}^{\widetilde{z}} d\ln\sigma(\tau-z) \\
&- m\int\limits_{z_1}^{z_1+2\omega} d\ln\sigma(\tau-z) - m'\int\limits_{z_1}^{z_1+2\omega'} d\ln\sigma(\tau-z)\Big\} \\
&= \widehat{\varphi}(z)\exp\Big\{\frac{1}{2\pi i}\int\limits_L \ln G(\tau)\zeta(\tau-z)d\tau - \ln\frac{\sigma(z-\widetilde{z})}{\sigma(z-z_1)} \\
&- m\frac{\sigma(z-z_1-2\omega)}{\sigma(z-z_1)} - m'\frac{\sigma(z-z_1-2\omega')}{\sigma(z-z_1)}\Big\} \\
&= \widehat{\varphi}(z)\exp\Big\{\frac{1}{2\pi i}\int\limits_L \ln G(\tau)\zeta(\tau-z)d\tau - \ln\frac{\sigma(z-\widetilde{z})}{\sigma(z-z_1)} \\
&- m\ln\frac{e^{-2\zeta(\omega)(z-z_1-2\omega)}\sigma(z-z_1)}{\sigma(z-z_1)} - m'\ln\frac{e^{-2\zeta(\omega')(z-z_1-2\omega')}\sigma(z-z_1)}{\sigma(z-z_1)}\Big\} \\
&= \varphi_1(z)\frac{\sigma(z-z_1)}{\sigma(z-\widetilde{z})}\exp\Big\{\frac{1}{2\pi i}\int\limits_L \ln G(\tau)\zeta(\tau-z)d\tau - 2m\zeta(\omega)(z-z_1) \\
&- 2m'\zeta(\omega')(z-z_1)\Big\} = \varphi_1(z)\frac{\sigma(z-z_1)}{\sigma(z-\widetilde{z})}e^{-2[m\zeta(\omega)-m'\zeta(\omega')]z} \\
&\times\exp\Big\{\frac{1}{2\pi i}\int\limits_L \ln G(\tau)\zeta(\tau-z)d\tau\Big\}.
\end{aligned}$$

Therefore the general solution of the problem (1.2) can be represented in the form

$$\Phi(z) = \varphi_1(z)\frac{\sigma(z-z_1)}{\sigma(z-\widetilde{z})}e^{-2[m\zeta(\omega)-m'\zeta(\omega')]z}\exp\Big\{\frac{1}{2\pi i}\int\limits_L \ln G(\tau)\zeta(\tau-z)d\tau\Big\}. \tag{3.4}$$

where $\varphi_1(z)$ is an arbitrary elliptic function, multiply to the given divisor $\mathcal{J}^{-1}\mathcal{D}^{-1}\mathcal{E}^{-1}\mathcal{F}^{-1}$. Here divisors $\mathcal{J}$ and $\mathcal{D}$ are those fixed in (1.2), the divisor $\mathcal{F}$ is defined from the solution of the Jacobi inversion problem, and the divisor $\mathcal{E}$ of the order $\varkappa$ is determined at the construction of the function χ in Section

2. The solution of the associated problem (1.3) has the form

$$\Psi(z) = \psi_1(z)\frac{\sigma(z-\tilde{z})}{\sigma(z-z_1)}e^{2[m\zeta(\omega)-m'\zeta(\omega')]z}\exp\Big\{-\frac{1}{2\pi i}\int\limits_L \ln G(\tau)\zeta(\tau-z)d\tau\Big\}, \tag{3.5}$$

where $\psi_1(z)$ is an arbitrary elliptic function multiply to the divisor $\mathcal{JEDF}$. Let l be a number of linear independent solutions of the homogeneous problem (1.2) (this number is equal to the dimension of the space of all elliptic functions multiply to the divisor $\mathcal{J}^{-1}\mathcal{D}^{-1}\mathcal{E}^{-1}\mathcal{F}^{-1}$). Let l' be a number of linear independent solutions of the homogeneous problem (1.3) (it is equal to the dimension of the space of all elliptic functions multiply to the divisor $\mathcal{JEDF}$). The following result is a straightforward consequence of the Riemann-Roch theorem and the Clifford theorem for the case $h = 1$ (see [2]):

Theorem 3.1. *The numbers l and l' satisfy the relation*

$$l - l' = \varkappa + m + n\ .$$

In particular, if $\varkappa + m + n > 0$ then $l = \varkappa + m + n$, $l' = 0$, if $\varkappa + m + n < 0$ then $l = 0$, $l' = -\varkappa - m - n$. In «the critical case» $\varkappa + m + n = 0$ the exact estimate $0 \leqslant l \leqslant 1$ holds.

If $\varkappa + m + m' = 0$ then two cases are possible, namely: $l = l' = 0$ or $l = l' = 1$. The last situation takes place if and only if $\tilde{z} = z_1$ is the solution of the equation (3.3). Then the formulas (3.4), (3.5) can be simplified:

$$\Phi(z) = \varphi_1(z)e^{-2[m\zeta(\omega)-m'\zeta(\omega')]z}\exp\Big\{\frac{1}{2\pi i}\int\limits_L \ln G(\tau)\zeta(\tau-z)d\tau\Big\}, \tag{3.6}$$

$$\Psi(z) = \psi_1(z)e^{2[m\zeta(\omega)-m'\zeta(\omega')]z}\exp\Big\{-\frac{1}{2\pi i}\int\limits_L \ln G(\tau)\zeta(\tau-z)d\tau\Big\}. \tag{3.7}$$

A general solution of the inhomogeneous problem (1.1) is the sum of a particular solution of (1.1) and the general solution of the homogeneous problem (1.2). Solvability conditions of the inhomogeneous problem (1.1) has the same form as in [5], where the following theorem was proved.

Theorem 3.2. *The necessary and sufficient solvability conditions of the problem* (1.1) *are the equalities*

$$\int\limits_L g(\tau)\Psi^+(\tau)d\tau = 0, \tag{3.8}$$

which are valid for all linear independent solutions $\Psi(z)$ of the problem (1.3).

Hence if $l' = 0$ then the problem (1.1) is unconditionally solvable. If $l' = 1$ then there exists one solvability condition (3.8).

Further we will use different analytic expressions for a particular solution of (1.1). Their choice depends on the value of l'. For the construction of a particular solution we must firstly factorize the coefficient G. If $l \geqslant 1$ then we can use representation (3.4) for such factorization. If $l' \geqslant 1$ then we can use representation (3.5). If $l = l' = 0$ then both formulas can be used with $\varphi_1(z) \equiv 1$ (respectively, $\psi_1(z) \equiv 1$).

Let, $l = \varkappa + m + m' \geqslant 1$ and Φ_0 be a particular solution of the homogeneous problem (1.2) contained in (3.4). Since zeroes of the solution (3.4) are arbitrary distributed then zeroes of Φ_0 can be fixed in such a way that the points of the divisor (Φ_0) lay outside of the contour L. It is possible if $l > 1$.

If $l = 1$ then it is possible that $a \in L$. In this case we can use another meromorphic analog of Cauchy kernel with characteristic divisor $(a)(b)^{-1}$ (see [3]):

$$\omega(z,\tau) = \frac{\sigma(\tau - z + a - b)}{\sigma(a-b)} \frac{\sigma(z-b)\sigma(\tau-a)}{\sigma(\tau-b)\sigma(z-a)} \frac{d\tau}{\sigma(\tau - z)}, \tag{3.9}$$

where a, b are two different points in $\sqcap \setminus L$, $\sigma(\cdot)$ is the Weierstrass sigma-function. Factorizing the coefficient G and using the described particular solution Φ_0 we reduce the problem (3.1) to the following "jump" problem:

$$\frac{\Phi^+(t)}{\Phi_0^+(t)} - \frac{\Phi^-(t)}{\Phi_0^-(t)} = \frac{g(t)}{\Phi_0^+(t)}, \quad t \in L, \tag{3.10}$$

where the new unknown function can have a pole at the point a. The problem (3.10) is unconditionally solvable and its solution can be found in the form of the Cauchy type integral with kernel (3.9). Hence, the particular solution of problem (1.1) is delivered by the formula

$$\Phi_1(z) = \frac{\Phi_0(z)}{2\pi i} \int\limits_L \frac{g(\tau)}{\Phi_0^+(\tau)} \omega(z,\tau) d\tau, \tag{3.11}$$

where ω is given in (3.9).

Let $l' \geqslant 1$, and Ψ_0 be a nontrivial solution of the associated problem (1.3) given in (3.5). Using Ψ_0 we factorize the coefficient G and reduce the boundary condition (1.1) to the following «jump» condition

$$\Phi^+(t)\Psi_0^+(t) - \Phi^-(t)\Psi_0^-(t) = g(t)\Psi_0^+(t), \quad t \in L. \tag{3.12}$$

Under condition (3.8) the solution of (3.12) can be represented in the form of the Cauchy type integral with the kernel $\zeta(\tau - z)d\tau$. Then a particular

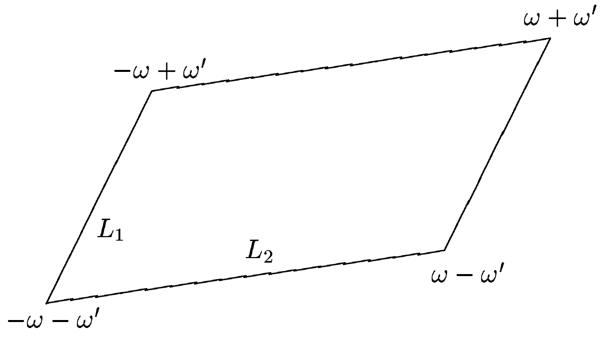

Figure 2. The contour of the region L.

solution of the inhomogeneous problem (1.1) can be delivered by the formula

$$\Phi_1(z) = \frac{1}{\Psi_0(z)}\left[C + \frac{1}{2\pi i}\int\limits_L g(\tau)\Psi_0^+(\tau)\zeta(\tau - z)d\tau\right],$$

where the constant C is determined from the condition that the expression in braces is multiply to the divisor $(\Phi_0)\mathcal{J}^{-1}\mathcal{D}^{-1}$.

Let now $l = l' = 0$. Then the coefficient G should be factorized by using the right-hand side of (3.4) with $\varphi_1(z) \equiv 1$. Denoting the solution (3.4) corresponding to $\varphi_1(z) \equiv 1$ we find a particular solution of (1.1) in the form

$$\Phi(q) = \frac{\chi_0(q)}{2\pi i}\int\limits_L \frac{g(\tau)}{\chi_0^+(\tau)}\omega_1(\tau, z)d\tau,$$

where $\omega_1(\tau, z)$ is the meromorphic analog of the Cauchy kernel with characteristic divisor $\mathcal{JEDF}$.

Example 3.1. Let consider the following problem:

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \ t \in L \tag{3.13}$$

where

$$G(t) = \begin{cases} G_1 & \text{if } t \in L_1, \\ G_2 & \text{if } t \in L_2\ , \end{cases}$$

and $G_1, G_2 \neq 0$ are given constants, and the contour $L = L_1 \bigcup L_2$ is presented in the Fig.2. Let us begin our analysis with the corresponding homogeneous problem

$$\Phi^+(t) = G(t)\Phi^-(t), \quad t \in L. \tag{3.14}$$

Applying the Argument Principle we can conclude that the solution of (3.14) (if any) has no zeroes on $\sqcap$. Hence by the monodromy theorem there exists on $\sqcap \setminus L$ a single-valued branch of the function $\ln \Phi(p)$. Therefore the condition (3.14) can be rewritten in the equivalent form

$$\ln \Phi^+(t) = \ln \Phi^-(t) + \ln G(t), \quad t \in L.$$

Differentiating the last equality we arrive at the condition

$$d \ln \Phi^+(t) = d \ln \Phi^-(t), \quad t \in L.$$

It follows from the analytic continuation theorem that $d \ln \Phi(p)$ is an abelian differential of the first kind, namely

$$d \ln \Phi(p) = C dz.$$

Hence the general solution of (3.14) can be found among the following ones

$$\Phi(p) = A \exp\{Cz\},$$

where A is an arbitrary constant.

Since $h = 1$, $\varkappa = 0$, $n = 0$, $m = 0$, then general solution of the homogeneous boundary value problem is delivered by the formula

$$\Phi(z) = C \exp\Big\{\frac{\ln G_1}{2\pi i} \int\limits_{L_1} \zeta(\tau - z) d\tau - \frac{\ln G_2}{2\pi i} \int\limits_{L_2} \zeta(\tau - z) d\tau\Big\}. \tag{3.15}$$

Using relation (3.2), we get the doubly periodicity condition of the right-hand side of (3.15):

$$\omega \ln G_1 + \omega' \ln G_2 = 0.$$

Let $\omega = \omega_1 + i\omega_2$ and $\omega' = \omega_1' + i\omega_2'$ then the latter condition is equivalent to the following one:

$$(\omega_1 + i\omega_2)(\ln |G_1| + i\mathrm{Arg} G_1) + (\omega_1' + i\omega_2')(\ln |G_2| + i\mathrm{Arg} G_2) = 0.$$

Taking real and imaginary parts of this equality we arrive at the following system of two real valued equations:

$$\begin{cases} \omega_1 \mathrm{Arg}\, G_1 \omega_2' + \mathrm{Arg}\, G_2 = \omega_1 \ln \big|G_1\big| - \omega_1' \ln \big|G_2\big|, \\ \omega_1 \mathrm{Arg}\, G_1 + \omega_1' \mathrm{Arg}\, G_1 = -(\omega_2 \ln \big|G_1\big| + \omega_2' \ln \big|G_2\big|). \end{cases}$$

Solving this system with respect to $\operatorname{Arg} G_1$ and $\operatorname{Arg} G_2$, we get

$$\operatorname{Arg} G_1 \equiv \frac{\triangle_1}{\triangle}\ (mod\, 2\pi), \quad \operatorname{Arg} G_2 \equiv \frac{\triangle_2}{\triangle}\ (mod\, 2\pi), \tag{3.16}$$

where

$$\begin{aligned}
\triangle &= \begin{vmatrix} \omega_2 & \omega_2' \\ \omega_1 & \omega_1' \end{vmatrix} = \omega_2\omega_1' - \omega_1\omega_2', \\
\triangle_1 &= \begin{vmatrix} \omega_1 \ln|G_1| + \omega_1' \ln|G_2| & \omega_2' \\ -(\omega_2 \ln|G_1| + \omega_2' \ln|G_2|) & \omega_1' \end{vmatrix} \\
&= \ln|G_1|(\omega_1\omega_1' + \omega_2\omega_2') + \ln|G_2|(\omega_1'^2 + \omega_1'^2), \\
\triangle_2 &= \begin{vmatrix} \omega_2 & \omega_1 \ln|G_1| + \omega_1' \ln|G_2| \\ \omega_1 & -(\omega_2 \ln|G_1| + \omega_2' \ln|G_2|) \end{vmatrix} \\
&= -(\ln|G_1|(\omega_2^2 + \omega_2'^2) + \ln|G_2|(\omega_2\omega_2' + \omega_1\omega_1')).
\end{aligned}$$

Under conditions (3.16) the general solution of the homogeneous problem (3.14) has the form

$$\begin{aligned}
\Phi(z) &= C \exp\left\{ \frac{\ln G_1}{2\pi i} \frac{\sigma(z-\omega+\omega')}{\sigma(z-\omega-\omega')} - \frac{\ln G_2}{2\pi i} \frac{\sigma(z+\omega+\omega')}{\sigma(z-\omega+\omega')} \right\} \\
&= C_1 \exp\left\{ \frac{\zeta(\omega')\ln G_1 - \zeta(\omega)\ln G_2}{\pi i} z \right\}
\end{aligned}$$

and the general solution of the associated problem is equal to

$$\Psi(z) = C_1 \exp\left\{ -\frac{\zeta(\omega')\ln G_1 - \zeta(\omega)\ln G_2}{\pi i} z \right\}.$$

In our case $l - l' = 0$. Therefore two variants are possible, namely $l = l' = 0$ and $l = l' = 1$. If $l = l' = 1$, then the inhomogeneous problem (3.13) has a solution if and only if the following condition

$$\int_L g(\tau)\Psi^+(\tau)d\tau = 0 \tag{3.17}$$

is satisfied. Here $\Psi(z)$ is the solution of the homogeneous associated problem. If (3.17) is valid then a particular solution of the problem (3.13) is given by the formula

$$\Phi_1(z) = \frac{\Phi_0(z)}{2\pi i} \int_L \frac{g(\tau)}{\Phi_0^+(\tau)} \zeta(\tau - z) d\tau,$$

where $\Phi_0(z)$ is a particular solution of the homogeneous problem (3.14), given by (3.16).

If $l = l' = 0$ then a meromorphic analogue of the Cauchy kernel can be taken in the form (3.9) with characteristic divisor $(a)(b)^{-1}$, where a , $b \in \sqcap\setminus L$, $a \neq b$. Then a particular solution of the inhomogeneous problem (3.13) can be presented in the form

$$\Phi_1(z) = \frac{\Phi_0(z)}{2\pi i} \int\limits_L \frac{g(\tau)}{\Phi_0^+(\tau)} \omega(z,\tau) d\tau.$$

REFERENCES

[1] N.I. Akhiezer. *Elements of the Theory of Elliptic Functions.* M., Nauka, 1970.

[2] N.G. Chebotarev. *Theory of Algebraic Functions.* M.-L., Gostekhizdat, 1948.

[3] N.A. Degtyrenko. Doubly periodic meromorpfic analog of the Cauchy kernel and its certain applications. *Izv. vyzov. Matematika*, **8(447)**, 11– 19, 1999.

[4] F.D. Gakhov. *Boundary Value Problems.* M., Nauka, 1977 (3-rd ed.).

[5] E.I. Zverovich. Boundary value problems of the theory of analytic functions in Hölder classes on the Riemann surfaces. *Uspekhi Mat. Nauk.*, **XXVI**(1(157)), 113 – 179, 1971.

Rymano kraštinio uždavinio dukart periodinėms funkcijoms išreikštinis sprendinys sudėtinio kontūro atveju

T.I. Gatalskaja

Straipsnyje išspręstas Rymano kraštinis uždavinys srityje su sudėtiniu kontūru. Rastas biperiodinis uždavinio sprendinys, išreikštas integralu su Košy branduoliu, kuris šiuo atveju yra Vajerštraso dzeta funkcija. Pabaigoje pateiktas iliustruojantis pavyzdys.