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# A Numerical Study on the Difference Solution of Singularly Perturbed Semilinear Problem with Integral Boundary Condition

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**Abstract.** The present study is concerned with the numerical solution, using finite difference method on a piecewise uniform mesh (Shishkin type mesh) for a singularly perturbed semilinear boundary value problem with integral boundary condition. First we discuss the nature of the continuous solution of singularly perturbed differential problem before presenting method for its numerical solution. The numerical method is constructed on piecewise uniform Shishkin type mesh. We show that the method is first-order convergent in the discrete maximum norm, independently of singular perturbation parameter except for a logarithmic factor. We give effective iterative algorithm for solving the nonlinear difference problem. Numerical results which support the given estimates are presented.

**Keywords:** singular perturbation, exponentially fitted difference scheme, Shishkin mesh, uniformly convergence, integral boundary condition.

**AMS Subject Classification:** 34D15; 65L05; 65L12; 65L20; 65L70.

## 1 Introduction

Differential equations to a class in which the highest derivative is multiplied by a small parameter are called singularly perturbed differential equations. The solutions of such equations typically contain layers which occur in narrow layer regions of the domain. This kind of problems arise very frequently in the fields of applied mathematics and physics which include fluid dynamics, quantum mechanics, elasticity, chemical reactions, gas porous electrodes theory, the Navier-Stokes equations of fluid flow at high Reynolds number, oceanography, meteorology, reaction-diffusion processes etc. It is well known that these problems depend on a small positive parameter $\varepsilon$ in such a way that the solution exhibits a multiscale character, i.e., there are thin transition layers where the solutions varies very rapidly for small values of $\varepsilon$, while away from layers it behaves regularly and varies slowly. Hence, the presence of small parameter in singularly perturbed problems presents severe difficulties that have to be addressed to ensure accurate numerical solutions [5, 17, 22, 23, 24, 30].

It is well known that singularly perturbed problems cannot be solved numerically by classical numerical methods with satisfactory results. Therefore it is important to develop suitable numerical methods for solving these problems, whose accuracy does not depend on the value of parameter $\varepsilon$, that is, methods that are convergent $\varepsilon$-uniformly. To solve these type of problems, mainly there are three approaches namely, fitted finite difference methods, finite element methods using special elements such as exponential elements, and fitted mesh methods which use a priori refined or special piecewise uniform grids which condense in the boundary layers in a special manner. One of the simplest ways to derive parameter-uniform methods consists of using a class of special piecewise uniform meshes, such as Shishkin type meshes (see [12, 13, 18, 19, 20] for the motivation for this type of mesh), which are constructed a priori and depend on the parameter $\varepsilon$, the problem data, and the number of corresponding mesh points. For the past two decades extensive researches have been made on numerical methods for solving singularly perturbed problems, see [10, 11, 12, 13, 14, 15, 18, 19, 20, 21, 25, 26, 27, 28, 29] and the references therein.

Differential equations with conditions which connect the values of the unknown solution at the boundary with values in the interior are known as nonlocal boundary value problems. Such problems arise in problems of semiconductors [15], in problems of hydromechanics [25], and some other physical phenomena. The problems with integral nonlocal conditions can be met in studying heat transfer problems [15, 25]. It have been studied extensively in the literature (see [10, 11, 14, 15, 21, 25, 27, 28, 29] and the references therein). Existence and uniqueness of the solutions of such problems can be found in [1, 4, 6, 16]. A linear version of the problem (1.1)-(1.2) has been studied in [8], where a finite difference scheme on an uniform mesh for solving singularly perturbed problem with integral nonlocal condition has been presented. It is well known that the difference schemes on a uniform mesh are not generally suitable to nonlinear singularly perturbed problems as a special fine mesh is required in boundary layer region and comparatively much coarser mesh elsewhere. Ideally, the mesh should be adapted to the features of the exact solution using an adaptive grid generation technique. This approach is now widely used for numerical solution of differential equations with steep, continuous solutions. Some approaches for the numerical solutions of three-point singularly perturbed boundary value problems have been proposed in [3, 7, 9].

We consider the following singularly perturbed semilinear boundary value problem with integral boundary condition:

$$Lu := \varepsilon^2 u''(x) + \varepsilon a(x) u'(x) - f(x, u(x)) = 0, \quad 0 < x < \ell, \tag{1.1}$$

$$u(0) = A,$$

$$L_0 u := u(\ell) - \int_{\ell_0}^{\ell_1} g(x)\, u(x)\, dx = B, \quad 0 \le \ell_0 < \ell_1 \le \ell, \tag{1.2}$$

where $0 < \varepsilon << 1$ is the perturbation parameter, $A$ and $B$ are given constants, the functions $a(x) \geqslant 0$ and $f(x,u)$ are sufficiently smooth on $[0,\ell]$ and $[0,\ell]\times\mathbb{R}$, respectively, and $g(x)$ is a continuous function on $[\ell_0, \ell_1]$, moreover

$$0 < \beta_* \leq \frac{\partial f}{\partial u} \leq \beta^* < \infty.$$

The solution $u$ generally has boundary layers near $x = 0$ and $x = \ell$.

This paper is concerned with $\varepsilon$-uniform numerical methods for a singularly perturbed semilinear boundary value problem with integral boundary condition. The difference schemes are constructed by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form [2, 3, 7, 8, 9]. This method of approximation has the advantage that the schemes can be effectively applied also in the case when the original problem has a solution with certain singularities (presence of boundary layer, nonsmooth solutions, etc.). The plan of the paper is as follows: Some important properties of the exact solution of singularly perturbed semilinear nonlocal boundary value problem (1.1)–(1.2) are presented in Section 2. Finite difference schemes on a piecewise uniform Shishkin type mesh for problem (1.1)–(1.2) are described in Section 3. Convergence properties of the scheme are analyzed in Section 4. Uniform convergence is proved in the discrete maximum norm.The iterative algorithm for solving the discrete problem is formulated, and numerical results are given in Section 5. The paper ends in Section 6 with conclusion.

**Notations**: Throughout the paper, $C$ will denote a generic positive constant independent of $\varepsilon$ and the mesh parameter. For any continuous function $g(x)$ defined on the corresponding interval, we use the maximum norm $\|g\|_\infty = \max\limits_{[0,\ell]} |g(x)|$.

## 2 Some properties of the continuous problem

Here we give useful asymptotic estimates of the exact solution of the problem (1.1)–(1.2) that are needed in later sections.

**Lemma 1.** *Let $u(x)$ be the solution of the problem(1.1)–(1.2), $a \in C^1[0,l]$ and $\gamma = \int_{\ell_0}^{\ell_1} |g(x)|\, dx < 1$. Then the inequalities*

$$\|u\|_\infty \leqslant C_0, \tag{2.1}$$

*where $C_0 = (1-\gamma)^{-1}\left(|A| + |B| + \beta^{-1}\|F\|_\infty\right)$, $F(x) = f(x,0)$, $\|u\|_\infty = \max\limits_{[0,\ell]} |u(x)|$ and*

$$\left|u'(x)\right| \leqslant C\left\{1 + \frac{1}{\varepsilon}\left(\exp(-\frac{c_0 x}{\varepsilon}) + \exp(-\frac{c_1(\ell - x)}{\varepsilon})\right)\right\}, 0 \leqslant x \leqslant \ell \tag{2.2}$$

*hold for the solution $u(x)$ provided that $\partial f/\partial u - \varepsilon a'(x) \geq \beta_*$ and $|\partial f/\partial x| \leqslant C$ for $x\in [0,\ell]$ and $|u| \leqslant C_0$, where*

$$c_0 = \frac{1}{2}\left(\sqrt{a^2(0) + 4\beta_*} + a(0)\right), \quad c_1 = \frac{1}{2}\left(\sqrt{a^2(\ell) + 4\beta_*} - a(\ell)\right).$$

*Proof.* We rewrite the problem (1.1)–(1.2) in the form

$$Lu := \varepsilon^2 u''(x) + \varepsilon a(x) u'(x) - b\,(x)\,u\,(x) = F\,(x)\,, \quad 0 < x < \ell, \tag{2.3}$$

$$u(0) = A, \quad L_0 u := u\,(\ell) - \int_{\ell_0}^{\ell_1} g\,(x)\,u\,(x)\,dx = B, \tag{2.4}$$

where

$$b\,(x) = \frac{\partial f}{\partial u}\,(x, \xi u\,(x))\,, \quad 0 < \xi < 1.$$

Here we use the maximum principle: Let $L$ and $L_0$ be the differential operators in (2.3)–(2.4) and $v \in C^2\,[0,\ell]$ . If $v\,(0) \geq 0$, $L_0 v \geq 0$ and $Lv \leq 0$ for all $0 < x < \ell$, then $v\,(x) \geq 0$ for all $0 \leq x \leq \ell$. Then from (2.3)–(2.4), using the maximum principle, we have the inequality

$$|u\,(x)| \leq |A| + |u\,(\ell)| + \beta^{-1} \|F\|_\infty\,, \quad x \in [0,\ell]\,. \tag{2.5}$$

Next, from boundary condition (2.4), we have

$$|u\,(\ell)| \leq |B| + \int_{\ell_0}^{\ell_1} |g\,(x)|\,|u\,(x)|\,dx. \tag{2.6}$$

By setting (2.6) in inequality (2.5), we obtain

$$\begin{aligned} |u\,(x)| &\leq |A| + |B| + \int_{\ell_0}^{\ell_1} |g\,(x)|\,|u\,(x)|\,dx + \beta^{-1} \|F\|_\infty \\ &\leq |A| + |B| + \max_{[\ell_0,\ell_1]} |u\,(x)| \int_{\ell_0}^{\ell_1} |g\,(x)|\,dx + \beta^{-1} \|F\|_\infty \\ &\leq |A| + |B| + \|u\|_\infty \int_{\ell_0}^{\ell_1} |g\,(x)|\,dx + \beta^{-1} \|F\|_\infty\,, \end{aligned}$$

which proves (2.1). The proof of (2.2) is almost identical to that of [9]. □

## 3 Discretization and layer - adapted mesh

In this section, we discretize problem (1.1)–(1.2) using a finite difference method on a piecewise uniform mesh of Shishkin type. The Shishkin mesh appropriate to this problem is introduced as follows.

### 3.1 Construction of the mesh

The approximation to the solution $u$ of problem (1.1)–(1.2) will be computed on a Shishkin mesh. This mesh is a piecewise uniform mesh, which is condensed in the boundary layer regions at $x = 0$ and $x = \ell$. For a divisible by 4 positive integer $N$, we divide the interval $[0,\ell]$ into the three subintervals $[0,\sigma_1]$, $[\sigma_1, \ell - \sigma_2]$ and $[\ell - \sigma_2, \ell]$, where transition points $\sigma_1$ and $\sigma_2$ are chosen such that

$$\sigma_1 = \min\left\{\ell/4, c_0^{-1}\varepsilon \ln N\right\}, \quad \sigma_2 = \min\left\{\ell/4, c_1^{-1}\varepsilon \ln N\right\},$$

where $c_0$ and $c_1$ are given in Lemma 2.1 and $N$ is the number of discretization points.

We divide each of the subinterval $[0,\sigma_1]$ and $[\ell-\sigma_2,\ell]$ into $\frac{N}{4}$ equidistant subinterval, while we divide the subinterval $[\sigma_1,\ell-\sigma_2]$ into $\frac{N}{2}$ equidistant subinterval. In practice, one usually has $\sigma_i << \ell$ $(i=1,2)$, so the mesh is fine on $[0,\sigma_1]$, $[\ell-\sigma_2,\ell]$ and coarse on $[\sigma_1,\ell-\sigma_2]$. We introduce the following notation for the three step-sizes:

$$h_1=\frac{4\sigma_1}{N},\quad h_2=\frac{2\left(\ell-\sigma_2-\sigma_1\right)}{N},\quad h_3=\frac{4\sigma_2}{N},$$
$$h_2+\frac{1}{2}\left(h_1+h_3\right)=\frac{2\ell}{N},\quad h_k\leq \ell N^{-1},\ k=1,3,\quad \ell N^{-1}\leq h_2\leq 2\ell N^{-1}.$$

We specify a set of mesh points $\bar{\omega}_N=\{x_i\}_{i=0}^{N}$,

$$x_i=\begin{cases} ih_1, & \text{for } i=0,1,2,...,N/4;\\ \sigma_1+(i-N/4)\,h_2, & \text{for } i=N/4+1,...,3N/4;\\ \ell-\sigma_2+(i-3N/4)\,h_3, & \text{for} i=3N/4+1,...,N.\end{cases}$$

### 3.2 Construction of the difference scheme

We introduce an arbitrary nonuniform mesh on the interval $[0,\ell]$

$$\omega_N=\{0<x_1<x_2<...<x_{N-1}<\ell\},\quad \bar{\omega}_N=\omega_N\cup\{x_0=0,x_N=\ell\}.$$

We set the step-size $h_i=x_i-x_{i-1}$, $i=1,2,...,N$. Before describing our numerical method, we introduce some notations for the mesh functions. We define the following finite difference for any mesh function $v_i=v(x_i)$ given on $\bar{\omega}_N$ :

$$v_i=v(x_i),\quad v_{\bar{x},i}=\frac{v_i-v_{i-1}}{h_i},\quad v_{x,i}=\frac{v_{i+1}-v_i}{h_{i+1}},\quad v_{\overset{0}{x},i}=\frac{v_{x,i}+v_{\bar{x},i}}{2},$$
$$v_{\widehat{x},i}=\frac{v_{i+1}-v_i}{\hbar_i},\quad v_{\bar{x}\widehat{x},i}=\frac{v_{x,i}-v_{\bar{x},i}}{\hbar_i},\quad \hbar_i=\frac{h_i+h_{i+1}}{2},$$
$$\|v\|_\infty\equiv\|v\|_{\infty,\bar{\omega}_N}:=\max_{0\leqslant i\leqslant N}|v_i|.$$

Our discretization for Eq. (1.1) will begin with the identity

$$\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}Lu(x)\varphi_i(x)dx=0,\ 1\leqslant i\leqslant N-1 \tag{3.1}$$

with the basis functions $\{\varphi_i(x)\}_{i=1}^{N-1}$ having the from

$$\varphi_i(x)=\begin{cases}\varphi_i^{(1)}(x), & x_{i-1}<x<x_i,\\ \varphi_i^{(2)}(x), & x_i<x<x_{i+1},\\ 0, & \text{otherwise},\end{cases}$$

where $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$, respectively, are the solutions of the following problems:

$$\varepsilon\varphi''-a_i\varphi'=0,\quad x_{i-1}<x<x_i,$$
$$\varphi\left(x_{i-1}\right)=0,\quad \varphi\left(x_i\right)=1,$$

$$\begin{aligned}&\varepsilon\varphi'' - a_i\varphi' = 0, \quad x_i < x < x_{i+1},\\ &\varphi(x_i) = 1, \quad \varphi(x_{i+1}) = 0.\end{aligned}$$

The functions $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$ can be explicitly expressed as following:

$$\begin{aligned}\varphi_i^{(1)}(x) &= \frac{e^{\frac{a_i(x-x_{i-1})}{\varepsilon}} - 1}{e^{\frac{a_ih_i}{\varepsilon}} - 1}, \qquad \varphi_i^{(2)}(x) = \frac{1 - e^{-\frac{a_i(x_{i+1}-x)}{\varepsilon}}}{1 - e^{-\frac{a_ih_{i+1}}{\varepsilon}}} \quad \text{for} \quad a_i \neq 0,\\ \varphi_i^{(1)}(x) &= \frac{x - x_{i-1}}{h_i}, \qquad \varphi_i^{(2)}(x) = \frac{x_{i+1} - x}{h_{i+1}} \quad \text{for} \quad a_i = 0.\end{aligned}$$

The coefficient $\chi_i$ in (3.1) is given by

$$\chi_i = \hbar_i^{-1}\int\limits_{x_{i-1}}^{x_{i+1}} \varphi_i(x)dx = \begin{cases} \hbar_i^{-1}\left(\frac{h_i}{1-e^{\frac{a_ih_i}{\varepsilon}}} + \frac{h_{i+1}}{1-e^{-\frac{a_ih_{i+1}}{\varepsilon}}}\right), & a_i \neq 0,\\ \qquad 1, & a_i = 0.\end{cases}$$

Rearranging (3.1) gives

$$\begin{aligned}-\varepsilon^2\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i'(x)u'(x)\,dx + \varepsilon a_i\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i(x)u'(x)\,dx\\ -f(x_i, u_i) + R_i = 0, \qquad i = 1, 2, ..., N-1 \qquad (3.2)\end{aligned}$$

with

$$\begin{aligned}R_i &= \varepsilon\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} [a(x) - a(x_i)]\,\varphi_i(x)u'(x)\,dx\\ &\quad - \chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} dx\varphi_i(x)\int_{x_{i-1}}^{x_{i+1}} \frac{d}{dx}f(\xi, u(\xi))K_{0,i}^*(x,\xi)\,d\xi, \qquad (3.3)\\ K_{0,i}^*(x,\xi) &= T_0(x-\xi) - T_0(x_i - \xi), \qquad i = 1, 2, ..., N-1,\\ T_0(\lambda) &= 1, \quad \lambda \geqslant 0, \quad T_0(\lambda) = 0, \quad \lambda < 0.\end{aligned}$$

Using the interpolating quadrature rules (2.1) and (2.2) from [2] with weight functions $\varphi_i(x)$ on subintervals $(x_{i-1}, x_{i+1})$ from (3.2), we obtain the following precise relation:

$$\begin{aligned}-\varepsilon^2\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i'(x)u'(x)\,dx + \varepsilon a_i\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i(x)u'(x)\,dx\\ = \varepsilon^2\left\{\chi_i^{-1}\left(1 + 0.5\varepsilon^{-1}\hbar_i a_i(\chi_{2,i} - \chi_{1,i})\right)\right\}u_{\bar{x}\hat{x},i} + \varepsilon a_i u_{\overset{0}{x},i},\end{aligned}$$

where

$$\chi_{1,i} = \hbar_i^{-1}\int\limits_{x_{i-1}}^{x_i} \varphi_i^{(1)}(x)dx = \begin{cases} \hbar_i^{-1}\left(\frac{\varepsilon}{a_i} + \frac{h_i}{1-e^{a_ih_i/\varepsilon}}\right), & a_i \neq 0,\\ \qquad \hbar_i^{-1}h_i/2, & a_i = 0,\end{cases}$$

$$\chi_{2,i} = \hbar_i^{-1}\int\limits_{x_i}^{x_{i+1}} \varphi_i^{(2)}(x)dx = \begin{cases} \hbar_i^{-1}\left(\frac{h_{i+1}}{1-e^{a_ih_{i+1}/\varepsilon}} - \frac{\varepsilon}{a_i}\right), & a_i \neq 0,\\ \qquad \hbar_i^{-1}h_{i+1}/2, & a_i = 0.\end{cases}$$

It then follows from this that

$$lu_i + R_i := \varepsilon\theta_i u_{\bar{x}\widehat{x},i} + \varepsilon a_i u_{\overset{0}{x},i} - f\left(x_i, u_i\right) + R_i = 0,\ 1 \leqslant i \leqslant N-1, \tag{3.4}$$

where

$$\theta_i = \chi_i^{-1}\left(1 + 0.5\varepsilon^{-1}\hbar_i a_i\left(\chi_{2,i} - \chi_{1,i}\right)\right). \tag{3.5}$$

After a simple calculation of (3.5), we obtain

$$\theta_i = \begin{cases} \frac{a_i\hbar_i}{2\varepsilon}\left(\frac{h_{i+1}\left(e^{\frac{a_ih_i}{\varepsilon}}-1\right)+h_i\left(1-e^{-\frac{a_ih_{i+1}}{\varepsilon}}\right)}{h_{i+1}\left(e^{\frac{a_ih_i}{\varepsilon}}-1\right)-h_i\left(1-e^{-\frac{a_ih_{i+1}}{\varepsilon}}\right)}\right), & a_i \neq 0, \\ 1, & a_i = 0. \end{cases} \tag{3.6}$$

Now, it remains to define an approximation for the boundary condition (1.2). Let $x_{N_0}$ and $x_{N_1}$ be the mesh points nearest to $\ell_0$ and $\ell_1$, respectively. Here we start with relations

$$\int_{\ell_0}^{\ell_1} g(x)\,u(x)\,dx = \int_{\ell_0}^{x_{N_0}} g(x)\,u(x)\,dx + \int_{x_{N_0}}^{x_{N_1}} g(x)\,u(x)\,dx + \int_{x_{N_1}}^{\ell_1} g(x)\,u(x)\,dx$$

$$\int_{x_{N_0}}^{x_{N_1}} g(x)\,u(x)\,dx = \sum_{i=N_0}^{N_1}\left(\int_{x_{i-1}}^{x_i} g(x)\,dx\right)u(x_i) + \bar{r}_i = S(u) + \bar{r}_i,$$

where

$$S(u) = \sum_{i=N_0}^{N_1}\left(\int_{x_{i-1}}^{x_i} g(x)\,dx\right)u(x_i), \tag{3.7}$$

$$\bar{r}_i = \sum_{i=N_0}^{N_1}\int_{x_{i-1}}^{x_i} dx\,g(x)\int_{x_{i-1}}^{x_i} u'(\xi)\left(T_0(x-\xi)-1\right)d\xi,$$

$$T_0(\lambda) = 1,\ \lambda \geqslant 0;\ T_0(\lambda) = 0,\ \lambda < 0.$$

Consequently

$$l_0 u := u(\ell) - S(u) = B + r, \tag{3.8}$$

where

$$r = \int_{\ell_0}^{x_{N_0}} g(x)\,u(x)\,dx + \int_{x_{N_1}}^{\ell_1} g(x)\,u(x)\,dx + \bar{r}_i. \tag{3.9}$$

Neglecting $R_i$ and $r$ in (3.4) and (3.8), we propose the following difference scheme for approximating the problem (1.1)–(1.2):

$$lu_i := \varepsilon^2\theta_i y_{\bar{x}\widehat{x},i} + \varepsilon a_i y_{\overset{0}{x},i} - f\left(x_i, y_i\right) = 0, \quad 1 \leqslant i \leqslant N-1, \tag{3.10}$$

$$y_0 = A, \quad l_0 y := y(\ell) - S(y) = B, \tag{3.11}$$

where $\theta_i$ and $S(y)$ are given by (3.6) and (3.7).

## 4 Analysis of the method

To investigate the convergence of this method, note that the error function $z = y - u$, $x \in \bar{\omega}_N$ is the solution of the discrete problem

$$\varepsilon^2 \theta_i z_{\bar{x}\hat{x},i} + \varepsilon a_i z_{\overset{0}{x},i} - [f(x_i, y_i) - f(x_i, u_i)] = R_i, \quad 1 < i < N, \tag{4.1}$$

$$z_0 = 0, \quad z_N - S(z) = r, \tag{4.2}$$

where $R_i$ and $r$ are defined by (3.3) and (3.9), respectively.

**Lemma 2.** *Let $z_i$ be the solution (4.1)–(4.2). Then the estimate*

$$\|z\|_{\infty,\bar{\omega}_N} \leq C \left( \|R\|_{\infty,\omega_N} + |r| \right) \tag{4.3}$$

*holds.*

*Proof.* The problem (4.1)–(4.2) can be rewritten as

$$lz_i := \varepsilon^2 \theta_i z_{\bar{x}\hat{x},i} + \varepsilon a_i z_{\overset{0}{x},i} - b_i z_i = R_i, \quad 1 < i < N, \tag{4.4}$$

$$z_0 = 0, \quad l_0 z := z_N - S(z) = r, \tag{4.5}$$

where $b_i = \frac{\partial f}{\partial u}(x_i, \tilde{y}_i)$ and $\tilde{y}_i$ is intermediate point called for by the mean value theorem.

Here we use the discrete maximum principle: Let $l$ and $l_0$ be the finite-difference operators in (4.4)–(4.5). If $v$ is any mesh function defined on $\bar{\omega}_N$ such that $v_0 \geq 0$, $l_0 v \geq 0$ and $lv_i \leq 0$ for all $i = 1, 2, ..., N-1$, then $v_i \geq 0$ for all $i = 0, 1, ..., N$.

According to the discrete maximum principle, we have the inequality

$$\|z\|_{\infty,\bar{\omega}_N} \leq \beta^{-1} \|R\|_{\infty,\omega_N} + |z_N|. \tag{4.6}$$

Next, from boundary condition (4.5), we get

$$|z_N| \leq |r| + \sum_{i=N_0}^{N_1} \left( \int_{x_{i-1}}^{x_i} |g(x)| \, dx \right) |z_i|. \tag{4.7}$$

By setting the inequality (4.7) in (4.6), we obtain

$$\begin{aligned} \|z\|_{\infty,\bar{\omega}_N} &\leq \beta^{-1} \|R\|_{\infty,\omega_N} + |r| + \sum_{i=N_0}^{N_1} \left( \int_{x_{i-1}}^{x_i} |g(x)| \, dx \right) |z_i| \\ &\leq \beta^{-1} \|R\|_{\infty,\omega_N} + |r| + \max_{N_0 \leq i \leq N_1} |z_i| \sum_{i=N_0}^{N_1} \int_{x_{i-1}}^{x_i} |g(x)| \, dx \\ &\leq \beta^{-1} \|R\|_{\infty,\omega_N} + |r| + \|z\|_{\infty,\bar{\omega}_N} \int_{\ell_0}^{\ell_1} |g(x)| \, dx. \end{aligned} \tag{4.8}$$

From (4.8), we have

$$\|z\|_{\infty,\bar{\omega}_N} \leq (1-\gamma)^{-1} \left( \|R\|_{\infty,\omega_N} + |r| \right).$$

Thus, since $\gamma < 1$, the estimate (4.3) follows. □

**Lemma 3.** *Under the assumptions of Section 1 and Lemma 1, the following estimates hold for the error functions $R_i$ and $r$ :*

$$\|R\|_{\infty,\omega_N} \le CN^{-1}\ln N, \quad |r| \le CN^{-1}\ln N, \tag{4.9}$$

*where $R_i$ and $r$ are defined by (3.3) and (3.9), respectively.*

*Proof.* From the explicit expression (3.3) for $R_i$, on an arbitrary mesh we get

$$|R_i| \le C\left\{h_i + h_{i+1} + \int_{x_{i-1}}^{x_{i+1}} (1+|u'(\xi)|)\, d\xi\right\}, \quad 1 \le i \le N.$$

This inequality, together with (2.2), enables us to write

$$|R_i| \le C\left\{h_i+h_{i+1}+\frac{1}{\varepsilon}\int_{x_{i-1}}^{x_{i+1}}\left(\exp(-\frac{c_0 x}{\varepsilon})+\exp(-\frac{c_1(\ell-x)}{\varepsilon}\right)dx\right\}. \tag{4.10}$$

In the first case we consider that $c_0^{-1}\varepsilon\ln N \ge \frac{\ell}{4}$ and $c_1^{-1}\varepsilon\ln N \ge \frac{\ell}{4}$, and the mesh is uniform with $h_1 = h_2 = h_3 = h = \ell N^{-1}$ for $1 \le i \le N$. Therefore, from (4.10) we obtain

$$\begin{aligned} |R_i| &\le C\left\{N^{-1}+\varepsilon^{-1}h\right\} \le C\left\{N^{-1}+4c_0^{-1}N^{-1}\ln N\right\} \\ &\le CN^{-1}\ln N,\ 1\le i\le N. \end{aligned}$$

In the second case we consider that $c_0^{-1}\varepsilon\ln N < \frac{\ell}{4}$ and $c_1^{-1}\varepsilon\ln N < \frac{\ell}{4}$, and the mesh is piecewise uniform with the mesh spacing $\frac{4\sigma_1}{N}$ and $\frac{4\sigma_2}{N}$ in the subintervals $[0,\ \sigma_1]$ and$[\ell-\sigma_2,\ \ell]$, respectively, and $\frac{2(\ell-\sigma_1-\sigma_2)}{N}$ in the subinterval $[\sigma_1,\ \ell-\sigma_2]$. We estimate $R_i$ on the subintervals $[0,\ \sigma_1]$, $[\sigma_1,\ \ell-\sigma_2]$, and $[\ell-\sigma_2,\ \ell]$ separately. In the layer region $[0,\ \sigma_1]$ the inequality (4.10) reduces to

$$|R_i| \le C\left(1+\varepsilon^{-1}\right)h_1 \le C\left(1+\varepsilon^{-1}\right)\frac{4c_0^{-1}\varepsilon\ln N}{N},\ 1\le i\le\frac{N}{4}-1.$$

Hence

$$|R_i| \le CN^{-1}\ln N,\ 1\le i\le\frac{N}{4}-1.$$

The same estimate is obtained in the layer region $[\ell-\sigma_2,\ \ell]$ in a similar manner. We now have to estimate $R_i$ for $\frac{N}{4}+1\le i\le\frac{3N}{4}-1$. In this case we are able to rewrite (4.10) as

$$\begin{aligned} |R_i| \le C\big\{&h_2+c_0^{-1}\left(\exp\left(-c_0x_{i-1}/\varepsilon\right)-\exp\left(-c_0x_{i+1}/\varepsilon\right)\right) \\ &+c_1^{-1}\left(\exp\left(-c_1\left(\ell-x_{i+1}\right)/\varepsilon\right)-\exp\left(-c_1\left(\ell-x_{i-1}\right)/\varepsilon\right)\right)\big\}, \\ N/4+1&\le i\le 3N/4-1. \end{aligned} \tag{4.11}$$

Since $x_i = c_0^{-1}\varepsilon\ln N+(i-N/4)\,h_2$ it follows that

$$\begin{aligned} &\exp\left(-c_0x_{i-1}/\varepsilon\right)-\exp\left(-c_0x_{i+1}/\varepsilon\right) \\ &\quad=\frac{1}{N}\exp\left(-c_0\left(i-1-\frac{N}{4}\right)h_2/\varepsilon\right)\left(1-\exp\left(-2c_0h_2/\varepsilon\right)\right)<N^{-1}. \end{aligned}$$

Also, if we rewrite the mesh points in the form $x_i = \ell - \sigma_2 - \left(\frac{3N}{4} - i\right) h_2$, evidently

$$\begin{aligned}&\exp\left(-c_1\left(\ell - x_{i+1}\right)/\varepsilon\right) - \exp\left(-c_1\left(\ell - x_{i-1}\right)/\varepsilon\right)\\&\quad= \frac{1}{N}\exp\left(-c_1\left(\frac{3N}{4} - i - 1\right)h_2/\varepsilon\right)\left(1 - \exp\left(-2c_1 h_2/\varepsilon\right)\right) < N^{-1}.\end{aligned}$$

The last two inequalities together with (4.11) give the bound

$$|R_i| \le CN^{-1}.$$

Finally, we estimate $R_i$ for the mesh points $x_{\frac{N}{4}}$ and $x_{\frac{3N}{4}}$. For the mesh point $x_{\frac{N}{4}}$, inequality (4.10) reduces to

$$\left|R_{\frac{N}{4}}\right| \le C\left\{\left(1 + \varepsilon^{-1}\right)h_1 + h_2 + \frac{1}{\varepsilon}\int_{x_{\frac{N}{4}}}^{x_{\frac{N}{4}+1}}\left(e^{-c_0 x/\varepsilon} + e^{-c_1(\ell - x)/\varepsilon}\right)dx\right\}.$$

Since

$$\exp\left(-\frac{c_0 x_{\frac{N}{4}}}{\varepsilon}\right) - \exp\left(-\frac{c_0 x_{\frac{N}{4}+1}}{\varepsilon}\right) = \frac{1}{N}\left(1 - \exp\left(-\frac{c_0 h_2}{\varepsilon}\right)\right) < N^{-1}$$

and

$$\begin{aligned}&\exp\left(-c_1\left(\ell - x_{\frac{N}{4}+1}\right)/\varepsilon\right) - \exp\left(-c_1\left(\ell - x_{\frac{N}{4}}\right)/\varepsilon\right)\\&\quad= \frac{1}{N}\exp\left(-c_1 h_1/\varepsilon\right)\left(1 - \exp\left(-c_1 h_1/\varepsilon\right)\right) < N^{-1},\end{aligned}$$

it then follows that

$$\left|R_{\frac{N}{4}}\right| \le CN^{-1}\ln N.$$

The same estimate is obtained for the mesh point $x_{\frac{3N}{4}}$ in a similar manner. This estimate is valid when only one of the values of $\sigma_1$ and $\sigma_2$ is equal to $\frac{\ell}{4}$. Thus the first inequality of estimate (4.9) is proved.

Next we estimate the remainder term $r$. From the explicit expression (3.9), we obtain

$$\begin{aligned}|r| &\le \sum_{i=N_0}^{N_1}\int_{x_{i-1}}^{x_i} dx\,|g(x)|\int_{x_{i-1}}^{x_i}|u'(\xi x)|\,|T_0(x-\xi) - 1|\,d\xi\\&\quad+ \int_{\ell_0}^{x_{N_0}}|g(x)|\,|u(x)|\,dx + \int_{x_{N_1}}^{\ell_1}|g(x)|\,|u(x)|\,dx\\&\le h_i\max_{[x_{i-1},\,x_i]}|g(x)|\sum_{i=N_0}^{N_1}\int_{x_{i-1}}^{x_i}|u'(\xi)|\,|T_0(x-\xi) - 1|\,d\xi + O(h_i)\\&\le 2h_i\max_{[x_{i-1},\,x_i]}|g(x)|\int_0^{\ell}|u'(x)|\,dx + O(h_i) \le Ch_i.\end{aligned}$$

If $[x_{N_0}, x_{N_1}]$ is inside the interval $[\sigma_1, \ell - \sigma_2]$, we obtain from the inequality (4.6)

$$|r| \leq CN^{-1}.$$

If $[x_{N_0}, x_{N_1}]$ is inside the interval $[0, \sigma_1]$, we deduce from the inequality (4.6) that

$$|r| \leq Ch^{(1)} \leq C\frac{4c_0^{-1}\varepsilon \ln N}{N} \leq CN^{-1}\ln N.$$

The same estimate is obtained for the interval $[\ell - \sigma_2, \ell]$ in a similar manner. This completes the proof of Lemma. □

We now can statement the convergence result of this paper.

**Theorem 1.** *Assume that $a, f \in C^1[0, \ell]$. Let $u$ be the solution of (1.1)–(1.2) and $y$ be the solution of (3.10)–(3.11). Then, the following $\varepsilon$-uniform estimate satisfies*

$$\|y - u\|_{\infty, \bar{\omega}_N} \leq CN^{-1}\ln N.$$

## 5 Algorithm and numerical result

In this section we present some numerical results for the difference scheme (3.10)–(3.11) applied to the problem (1.1)–(1.2) on the piecewise uniform mesh of Shishkin type.

We solve the nonlinear problem (3.10)–(3.11) using the following quasilinearization technique:

$$\varepsilon^2\theta_i y^{(n)}_{\bar{x}\widehat{x},i} + \varepsilon a_i y^{(n)}_{\overset{0}{x},i} - f\left(x_i, y_i^{(n-1)}\right) - \frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right)\left(y_i^{(n)} - y_i^{(n-1)}\right) = 0,$$

$$y_0^{(n)} = A, \quad y_N^{(n)} = \sum_{i=N_0}^{N_1} h_i g_i y_i^{(n-1)} + B,$$

for $n \geq 1$ and $y_i^{(0)}$ given for $1 \leq i \leq N$.

*Example 1.* First we study the following test problem:

$$\varepsilon^2 u'' + \varepsilon\sin(\frac{\pi x}{2})u' + e^{-u} - x^2 = 0, 0 < x < 1,$$

$$u(0) = 1, \quad u(1) = \int_{0.5}^{1}\cos(x)u(x)\,dx + 1.$$

For this problem the exact solution is unknown. Therefore we use the double-mesh principle to estimate the errors and compute solutions, that is, we compare the computed solution with the solution on a mesh that is twice as fine (see [9, 12, 13]. The Table 1 shows our numerical results for the first problem. We measure the accuracy in the discrete maximum norm

$$e_\varepsilon^N = \max_i \left|y_i^{\varepsilon,N} - \tilde{y}_i^{\varepsilon,2N}\right|,$$

**Table 1.** Approximate errors $e_\varepsilon^N$ and the computed rates of convergence $P^N$ on $\omega_N$ for various values of $\varepsilon$ and $N$ $(a(0) = 0)$

| $\varepsilon$ | $N = 16$ | $N = 32$ | $N = 64$ | $N = 128$ | $N = 256$ | $N = 512$ |
|---|---|---|---|---|---|---|
| $2^{-2}$ | 0.056424 | 0.030112 | 0.001555 | 0.007963 | 0.004024 | 0.002068 |
| | 0.84 | 0.89 | 0.95 | 0.97 | 0.98 | |
| $2^{-4}$ | 0.058452 | 0.030321 | 0.001554 | 0.007951 | 0.004016 | 0.002035 |
| | 0.85 | 0.88 | 0.94 | 0.96 | 0.99 | |
| $2^{-6}$ | 0.059024 | 0.030213 | 0.001550 | 0.007864 | 0.004013 | 0.001996 |
| | 0.85 | 0.87 | 0.94 | 0.96 | 0.99 | |
| $2^{-8}$ | 0.059015 | 0.030124 | 0.001551 | 0.007834 | 0.004008 | 0.001996 |
| | 0.85 | 0.87 | 0.94 | 0.96 | 0.99 | |
| $2^{-10}$ | 0.059016 | 0.030985 | 0.001549 | 0.007829 | 0.004005 | 0.001995 |
| | 0.85 | 0.87 | 0.94 | 0.96 | 0.99 | |
| $2^{-12}$ | 0.059013 | 0.030988 | 0.001549 | 0.007827 | 0.004005 | 0.001996 |
| | 0.85 | 0.87 | 0.94 | 0.96 | 0.99 | |
| $2^{-14}$ | 0.059014 | 0.030986 | 0.001550 | 0.007827 | 0.004006 | 0.001996 |
| | 0.85 | 0.87 | 0.94 | 0.96 | 0.99 | |
| $2^{-16}$ | 0.059014 | 0.030987 | 0.001550 | 0.007828 | 0.004006 | 0.001996 |
| | 0.85 | 0.87 | 0.94 | 0.96 | 0.99 | |
| $e^N$ | 0.059024 | 0.030988 | 0.001555 | 0.007963 | 0.004024 | 0.002035 |
| $p^N$ | 0.85 | 0.89 | 0.95 | 0.97 | 0.99 | |

where $\tilde{y}_i^{\varepsilon,2N}$ is the approximate solution of the respective method on the mesh

$$\tilde{\omega}_{2N} = \left\{ x_{\frac{i}{2}} : i = 0, 1, 2, ..., 2N \right\}$$

with

$$x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2} \quad \text{for} \quad i = 0, 1, 2, ..., N - 1.$$

The rates of convergence are defined as

$$P_\varepsilon^N = \frac{\ln \left( e_\varepsilon^N / e_\varepsilon^{2N} \right)}{\ln 2}.$$

The $\varepsilon-$uniform errors $e^N$ are estimated from

$$e^N = \max_\varepsilon e_\varepsilon^N.$$

The corresponding $\varepsilon-$uniform the rates of convergence are computed using the formula

$$P^N = \frac{\ln \left( e^N / e^{2N} \right)}{\ln 2}.$$

*Example 2.* We consider the following our second test problem:

$$\varepsilon^2 u'' + \varepsilon (1 + x) u' - 2u + \arctan (x + u) = 0, \quad 0 < x < 1,$$

$$u(0) = 0, \quad u(1) + \int_{0.5}^{1} \cos(2x) u(x)\, dx = 1.$$

The exact solution of this problem is unknown. We therefore use the double-mesh principle in a similar way to the first example. The Table 2 shows our numerical results for the second problem.

**Table 2.** Approximate errors $e_\varepsilon^N$ and the computed rates of convergence $P^N$ on $\omega_N$ for various values of $\varepsilon$ and $N$ $(a(0) \neq 0)$

| $\varepsilon$ | $N=16$ | $N=32$ | $N=64$ | $N=128$ | $N=256$ |
|---|---|---|---|---|---|
| $2^{-2}$ | 0.0350188 | 0.0169087 | 0.0081966 | 0.0040208 | 0.0020104 |
| | 1.05 | 1.04 | 1.03 | 1.00 | |
| $2^{-4}$ | 0.0445136 | 0.0226725 | 0.0107762 | 0.0050468 | 0.0024715 |
| | 0.97 | 1.07 | 1.09 | 1.03 | |
| $2^{-6}$ | 0.0606108 | 0.0306246 | 0.0146222 | 0.0069817 | 0.0034190 |
| | 0.98 | 1.07 | 1.07 | 1.03 | |
| $2^{-8}$ | 0.0660523 | 0.0339593 | 0.0163933 | 0.0079434 | 0.0039443 |
| | 0.96 | 1.05 | 1.05 | 1.01 | |
| $2^{-10}$ | 0.0674129 | 0.0347566 | 0.0167837 | 0.0081368 | 0.0040403 |
| | 0.96 | 1.05 | 1.04 | 1.01 | |
| $2^{-12}$ | 0.0677529 | 0.0349534 | 0.0168777 | 0.0081819 | 0.0040626 |
| | 0.95 | 1.05 | 1.04 | 1.01 | |
| $2^{-14}$ | 0.0678379 | 0.0350025 | 0.0169009 | 0.0081929 | 0.0040681 |
| | 0.95 | 1.05 | 1.04 | 1.01 | |
| $2^{-16}$ | 0.0678592 | 0.0350147 | 0.0169067 | 0.0081957 | 0.0040695 |
| | 0.95 | 1.05 | 1.04 | 1.01 | |
| $e^N$ | 0.0678592 | 0.0350147 | 0.0169067 | 0.0081957 | 0.0040695 |
| $p^N$ | 0.95 | 1.04 | 1.03 | 1.01 | |

## Conclusions

We have presented a fitted finite difference method on the piecewise uniform mesh for solving singularly perturbed semilinear boundary value problem with integral nonlocal condition. The difference scheme is based on the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form. Our method has the advantage that the scheme can be effectively applied also in the case when the original problem has a solution with certain singularities. It is shown that the method is $\varepsilon$-uniform convergence with respect to the perturbation parameter in the discrete maximum norm. We have implemented the present method on two standard test problems. Using the double mesh, the computed maximum pointwise errors $e_\varepsilon^N$ and $e_\varepsilon^{2N}$, and the rates of uniform convergence $P_\varepsilon^N$ for different values of $\varepsilon$ and $N$ are presented in Tables 1 and 2. The rates of convergence $P_\varepsilon^N$ are monotonically increasing towards one. It is observed from the results that numerical experiments are in agreement with the theoretical results. The main lines for the analysis of the uniform convergence carried out here can be used for the study of more complicated linear differential problems as well as nonlinear differential problems with mixed nonlocal boundary conditions.

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