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STRONGLY (V^{λ}, A, P) – SUMMABLE SEQUENCE SPACES DEFINED BY A MODULUS

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Abstract. We introduce the strongly (V^{λ}, A, p) – summable sequences and give the relation between the spaces of strongly (V^{λ}, A, p) – summable sequences and strongly (V^{λ}, A, p) – summable sequences with respect to a modulus function when $A = (a_{ik})$ is an infinite matrix of complex numbers and $p = (p_i)$ is a sequence of positive real numbers. Also we give natural relationship between strongly (V^{λ}, A, p) – convergence with respect to a modulus function and strongly $S^{\lambda}(A)$ – statistical convergence.

Key words: De la Vallee-Poussin mean, modulus function, statistical convergence

1. Introduction

Let $\lambda = (\lambda_r)$ be a nondecreasing sequence of positive numbers tending to ∞ and $\lambda_{r+1} \leq \lambda_r + 1, \lambda_1 = 1$. The generalized de la Vallee-Poussin mean is defined by

$$t_r(x) = \lambda_r^{-1} \sum_{i \in I_r} x_i, \quad I_r = [r - \lambda_r + 1, r].$$

A sequence $x = (x_i)$ is said to be (V, λ) – summable to a number s if $t_r(x) \to s$ as $r \to \infty$ (Leindler [7]). If $\lambda_r = r$, then the (V, λ)– summability is reduced to $(C, 1)$ – summability. We write

$$[V, \lambda] = \{x = (x_i) : \lim_{r \to \infty} \lambda_r^{-1} \sum_{i \in I_r} |x_i - s| = 0 \ \text{ for some } s\}$$

for sets of sequences $x = (x_i)$ which are strongly (V, λ) – summable to s, that is $x_i \to s[V, \lambda]$.

Subsequently strongly (V,λ) – summable sequence spaces have been studied by various authors: (Bilgin [1], Güngör et al [5], Savas [14], and others). The notion of modulus function was introduced by Nakano [10]. We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $f(x)=0$ if and only if $x=0$,

(ii) $f(x+y)\le f(x)+f(y)$ for $x,y\ge 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

It follows that f must be continuous on $[0,\infty)$.

Bilgin [1], Kolk [6], Maddox [8, 9], Öztürk and Bilgin [11], Ruckle [12], and others used a modulus function for defining some new sequence spaces.

Let $A=(a_{ik})$ be an infinite matrix of complex numbers. We write $Ax=(A_i(x))$ if $A_i(x)=\sum_{k=1}^{\infty} a_{ik}x_k$ converges for each i.

Recently, the concept of strong (V,λ) – summability was generalized by Bilgin [1], as follows:

$$V^{\lambda}[A,f]=\{x=(x_i):\lim_{r\to\infty}\lambda_r^{-1}\sum_{i\in I_r} f(|A_i(x)-s|)=0,\ \text{ for some } s\}.$$

In the present paper we introduce the strongly (V^{λ},A,p) – summable sequences and give the relation between the spaces of strongly (V^{λ},A,p) – summable sequences and strongly (V^{λ},A,p) – summable sequences with respect to a modulus when $A=(a_{ik})$ is an infinite matrix of complex numbers and $p=(p_i)$ is a sequence of positive real numbers. Also we give natural relationship between strongly (V^{λ},A,p) – convergence with respect to a modulus function and strongly $S^{\lambda}(A)$ – statistical convergence.

The following inequality will be used throughout the paper;

$$|a_i+b_i|^{p_i}\le T(|a_i|^{p_i}+|b_i|^{p_i}) \tag{1.1}$$

where a_i and b_i are complex numbers, $T=\max(1,2^{H-1})$, and $H=supp_i<\infty$.

2. Strongly (V^{λ},A,p) – Summable Sequences

Let $A=(a_{ik})$ be an infinite matrix of complex numbers, $p=(p_i)$ be a bounded sequence of positive real numbers $(0<h=\inf p_i\le p_i\le supp_i=H<\infty)$, and f be a modulus. We define

$$V^{\lambda}[A,p,f]=\{x:\lim_{r\to\infty}\lambda_r^{-1}\sum_{i\in I_r} f(|A_i(x)-s|)^{p_i}=0,\ \text{for some } s\},$$

$$V_0^{\lambda}[A,p,f]=\{x:\lim_{r\to\infty}\lambda_r^{-1}\sum_{i\in I_r} f(|A_i(x)|)^{p_i}=0\},$$

$$V_\infty^{\lambda}[A,p,f]=\{x:\sup_r\lambda_r^{-1}\sum_{i\in I_r} f(|A_i(x)|)^{p_i}<\infty\}.$$

A sequence $x = (x_k)$ is said to be strongly (V^λ, A, p) – convergent to a number s with respect to a modulus if there is a complex number s such that $x \in V^\lambda[A, p, f]$. If x is strongly $((V^\lambda, A, p)$ – convergent to the value s with respect to a modulus f, then we write $x_i \to s(V^\lambda[A, p, f])$.

Throughout the paper μ will denote one of the notations $0, 1$ or ∞.

When $f(x) = x$, then we write the spaces $V_\mu^\lambda[A, p]$ in place of $V_\mu^\lambda[A, p, f]$. If $p_i = 1$ for all i, $V_\mu^\lambda[A, p, f]$ reduce to $V_\mu^\lambda[A, f]$. Hence $V_\mu^\lambda[A, p, f]$ is the same as the space $[A, V, \lambda, f]$ of Bilgin [1].

In this section we examine some topological properties of $V^\lambda[A, p, f]$ spaces and investigate some inclusion relations between these spaces.

Theorem 1. *Let f be a modulus function and X denotes the anyone of the spaces $V^\lambda[A, p, f]$, $V_0^\lambda[A, p, f]$ or $V_\infty^\lambda[A, p, f]$. Then X is a linear space over the complex field C.*

Proof. We give the proof only for $V_0^\lambda[A, p, f]$. Since the proof is analogous for the spaces $V^\lambda[A, p, f]$ and $V_\infty^\lambda[A, p, f]$, we omit the details.

Let $x, y \in V_0^\lambda[A, p, f]$, and $\alpha, \beta \in C$. Then there exist integers T_a and T_b such that $|a| \le T_a$ and $|b| \le T_b$. We therefore have

$$\lambda_r^{-1} \sum_{k \in I_r} f\big(\big|\sum_{k=1}^{\infty} a_{ik}(ax_k + by_k)\big|\big)^{p_k} \le \lambda_r^{-1} \sum_{k \in I_r} f\big(\big|\sum_{k=1}^{\infty} a_{ik} a x_k + \sum_{k=1}^{\infty} a_{ik} b y_k\big|\big)^{p_k}$$

$$\le T\{\lambda_r^{-1} \sum_{k \in I_r} [T_a f\big(\big|\sum_{k=1}^{\infty} a_{ik} x_k\big|\big) + \lambda_r^{-1} \sum_{k \in I_r} T_b f\big(\big|\sum_{k=1}^{\infty} a_{ik} y_k\big|\big)]^{p_k}$$

$$\le T\{[T_a]^H \lambda_r^{-1} \sum_{k \in I_r} f\big(\big|\sum_{k=1}^{\infty} a_{ik} x_k\big|\big)^{p_k} + [T_b]^H \lambda_r^{-1} \sum_{k \in I_r} f\big(\big|\sum_{k=1}^{\infty} a_{ik} y_k\big|\big)^{p_k}$$

as $r \to \infty$. This proves that $V_0^\lambda[A, p, f]$ is linear. ■

Theorem 2. *If f be any modulus, then the inclusions $V_0^\lambda[A, p, f] \subset V^\lambda[A, p, f] \subset V_\infty^\lambda[A, p, f]$ hold.*

Proof. The inclusion $V_0^\lambda[A, p, f] \subset V^\lambda[A, p, f]$ is obvious. Now let $x \in V^\lambda[A, p, f]$ such that $x_i \to s(\ V^\lambda[A, p, f])$. By using (1.1), we have

$$\sup_r \lambda_r^{-1} \sum_{i \in I_r} f(|A_i(x)|)^{p_i} = \sup_r \lambda_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s + s|)^{p_i}$$

$$\le T\{\sup_r \lambda_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|)^{p_i} + \sup_r \lambda_r^{-1} \sum_{i \in I_r} f(|s|)^{p_i}\}$$

$$\le T\{\sup_r \lambda_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|)^{p_i} + Max\{f(|s|)^h, f(|s|)^H\}\} < \infty.$$

Hence $x \in V_\infty^\lambda[A, p, f]$. This shows that the inclusion $V^\lambda[A, p, f] \subset V_\infty^\lambda[A, p, f]$ holds which completes the proof. ■

The proof of the following result is a consequence of Theorem 2

Corollary 1. $V_0^{\lambda}[A,p,f]$ and $V^{\lambda}[A,p,f]$ are nowhere dense subsets of $V_{\infty}^{\lambda}[A,p,f]$.

Let X be a sequence space. Then X is called

Solid (or normal) if $(\alpha_i x_i) \in X$ whenever $(x_i) \in X$ for all sequences (α_i) of scalars with $|\alpha_i| \leq 1$; for all $i \in N$;

Monotone provided X contains the canonical preimages of all its stepspaces.

If X is normal, then X is monotone.

Theorem 3. *The sequence spaces* $V_0^{\lambda}[A,p,f]$ *and* $V_{\infty}^{\lambda}[A,p,f]$ *are solid and hence monotone.*

Proof. Let $\alpha = (\alpha_i)$ be sequence of scalars such that $|\alpha_i| \leq 1$; for all $i \in N$. Since f is monotone, we get

$$\lambda_r^{-1} \sum_{i \in I_r} f(|A_i(\alpha x)|)^{p_i} \leq \lambda_r^{-1} \sum_{i \in I_r} f(\sup|\alpha_i|\,|A_i(x)|)^{p_i} \leq \lambda_r^{-1} \sum_{i \in I_r} f(|A_i(x)|)^{p_i},$$

which leads us to the desired result. ∎

Now we give relation between strongly (V^{λ}, A, p) – convergence and strongly (V^{λ}, A, p) – convergence with respect to a modulus.

Theorem 4. *Let* f *be any modulus. Then* $V_{\mu}^{\lambda}[A,p] \subseteq V_{\mu}^{\lambda}[A,p,f]$.

Proof. We consider only case $V_0^{\lambda}[A,p] \subseteq V_0^{\lambda}[A,p,f]$. Let $x \in V_0^{\lambda}[A,p]$ and $\varepsilon > 0$. We choose $0 < \delta < 1$ such that $f(u) < \varepsilon$ for every u with $0 \leq u \leq \delta$. We can write

$$\begin{aligned} \lambda_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|)^{p_i} &= \lambda_r^{-1} \sum_{1} f(|A_i(x) - s|)^{p_i} + \lambda_r^{-1} \sum_{2} f(|A_i(x) - s|)^{p_i} \\ &\leq \max(\varepsilon^h, \varepsilon) + \max(1, (2f(1)\delta^{-1})^H) \lambda_r^{-1} \sum_{2} f(|A_i(x) - s|)^{p_i} \end{aligned}$$

where the summation $\sum_1$ is over $|A_i(x) - s| \leq \delta$ and the summation $\sum_2$ is over $|A_i(x) - s| > \delta$. Hence

$$\frac{1}{\lambda_r} \sum_{i \in I_r} f(|A_i(x) - s|)^{p_i} \leq \max(\varepsilon^h, \varepsilon) + \max\left(1, \left(\frac{2f(1)}{\delta}\right)^H\right) \frac{1}{\lambda_r} \sum_{i \in I_r} |A_i(x) - s|^{p_i},$$

Therefore $x \in V_0^{\lambda}[A,p,f]$. ∎

Theorem 5. *Let* f *be any modulus. If* $\lim_{t \to \infty} \frac{f(t)}{t} = \beta > 0$, *then* $V_{\mu}^{\lambda}[A,p] = V_{\mu}^{\lambda}[A,p,f]$.

Proof. For any modulus function, the existence of a positive limit given with β was introduced by Maddox [9].

Let $\beta > 0$ and $x \in V_{\mu}^{\lambda}[A,p,f]$. Since $\beta > 0$, we have $f(t) \geq \beta t$ for all $t > 0$. From this inequality,it is easy to see that $x \in V_{\mu}^{\lambda}[A,p]$. By using Theorem 4 the proof is completed. ∎

We consider that (p_k) and $q_k)$ are any bounded sequences of strictly positive real numbers. We are able to prove $V_\mu^\lambda[A,q,f] \subset V_\mu^\lambda[A,p,f]$ only under additional conditions.

Theorem 6. *Let* $0 < p_i \leq q_i$ *for all* k *and let* (q_i/p_i) *be bounded. Then* $V_\mu^\lambda[A,q,f] \subset V_\mu^\lambda[A,p,f]$.

Proof. If we take $t_i = f(|A_i(x)|)^{q_i}$ for all i, then using the same technique employed in the proof of Theorem 2 of Öztürk and Bilgin [11], it is easy to prove the theorem. ■

Corollary 2. The following statements are valid:
(i) if $0 < \inf p_i \leq 1$ for all k, then $V_\mu^\lambda[A,f] \subset V_\mu^\lambda[A,p,f]$,
(ii) if $1 \leq p_i \leq \sup p_i = H < \infty$, then $V_\mu^\lambda[A,p,f] \subset V_\mu^\lambda[A,f]$.

Proof. (i) follows from Theorem 6 with $q_i = 1$ for all i and (ii) follows from Theorem 6 with $p_i = 1$ for all i. ■

3. $S^\lambda(A)$ – Statistical Convergence

In this section, we introduce natural relationship between strongly (V^λ, A, p) – convergence with respect to a modulus function and strongly $S^\lambda(A)$ – statistical convergence. In [3], Fasth introduced the idea of statistical convergence. These idea was later studied by Connor [2], Freedman and Sember [4], Salat [13], Savas [14], Schoenberg [15] and the other authors independently.

A complex number sequence $x = (x_i)$ is said to be statistically convergent to the number l if for every $\varepsilon > 0$, $\lim_{n\to\infty} |K(\varepsilon)/n| = 0$, where $|K(\varepsilon)|$ denotes the number of elements in $K(\varepsilon) = \{i \in N : |x_i - l| \geq \varepsilon\}$. The set of statistically convergent sequences is denoted by S.

A sequence $x = (x_i)$ is said to be strongly $S^\lambda(A)$– statistically convergent to s if any $\varepsilon > 0$, $\lim_{r\to\infty} \lambda_r^{-1} |KA(\varepsilon)| = 0$, where $|K(\varepsilon)|$ denotes the number of elements in $KA(\varepsilon) = \{i \in I_r : |A_i(x) - s| \geq \varepsilon\}$ The set of all strongly $S^\lambda(A)$ – statistically convergent sequences is denoted by $S^\lambda(A)$ (or $S(\lambda, A)$), Bilgin[1].

Now we give the relation between $S^\lambda(A)$ – statistical convergence and strongly (V^λ, A, p) – convergence with respect to a modulus.

Theorem 7. *Let* f *be a modulus function. Then* $V^\lambda[A,p,f] \subset S^\lambda(A)$.

Proof. Let $x \in V^\lambda[A,p,f]$. Then

$$\lambda_r^{-1} \sum_{i\in I_r} f(|A_i(x) - s|)^{p_i} \geq \lambda_r^{-1} \sum_1 f(|A_i(x) - s|)^{p_i} \geq \lambda_r^{-1} \sum_1 f(\varepsilon)^{p_i}$$

$$\geq \frac{1}{\lambda_r} \sum_1 \min\{f(\varepsilon)h, f(\varepsilon)^H\} \geq \frac{1}{\lambda_r} |\{i \in I_r : |A_i(x) - s| \geq \varepsilon\}| \min\{f(\varepsilon)^h, (\varepsilon)^H\},$$

where the summation $\sum_1$ is over $|A_i(x) - s| \geq \varepsilon$. Hence $x \in S^\lambda(A)$. ■

Theorem 8. *Let f be a bounded modulus function. Then $V^{\lambda}[A,p,f] = S^{\lambda}(A)$.*

Proof. By Theorem 7, it is sufficient to show that $S^{\lambda}(A) \subset V^{\lambda}[A,p,f]$. Let $x \in S^{\lambda}(A)$. Since f is bounded, so there exists an integer $K > 0$ such that $f(|A_i(x) - s|) \le K$. Then for a given $\varepsilon > 0$; we have

$$\lambda_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|)^{p_i} = \lambda_r^{-1} \sum_{1} f(|A_i(x) - s|)^{p_i} + \lambda_r^{-1} \sum_{2} f(|A_i(x) - s|)^{p_i}$$
$$\le K^H \lambda_r^{-1} \left|\{i \in I_r : |A_i(x) - s| \ge \varepsilon\}\right| + \max\{f(\varepsilon)^h, f(\varepsilon)^H\},$$

where the summation $\sum_1$ is over $|A_i(x) - s| \ge \varepsilon$ and the summation $\sum_2$ is over $|A_i(x) - s| < \varepsilon$. Taking the limit as $\varepsilon \to 0$ and $r \to \infty$, it follows that $x \in V^{\lambda}[A,p,f]$. This completes the proof. ■

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