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# ALTERNATING DIRECTION METHOD FOR A TWO-DIMENSIONAL PARABOLIC EQUATION WITH A NONLOCAL BOUNDARY CONDITION

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**Abstract.** The present paper deals with an alternating direction implicit method for a two dimensional parabolic equation in a rectangle domain with a nonlocal boundary condition in one direction. Sufficient conditions of stability for Peaceman-Rachford method are established. Results of some numerical experiments are presented.

**Key words:** parabolic equation, nonlocal boundary conditions, the Peaceman–Rachford method, alternating direction implicit method

## 1. Introduction

Theoretical analysis of differential problems with nonlocal boundary conditions (NBC) of various types and finite-difference methods for them often appeared in scientific literature n the past decade [3, 4, 7, 8, 10, 17, 18, 23]. Two-dimensional and three-dimensional problems are the focus of attention [16]. Economical finite difference schemes (FDS) and their theoretical investigation are one of the main issues of numerical mathematics [19]. Various algorithms are under consideration for solving 2D or 3D parabolic problems with the integral condition when a multidimensional case is reduced to 1D problems [5, 6, 9, 11, 12, 13, 14, 15].

This paper deals with the Peaceman-Rachford Alternating Direction Implicit (ADI) method [1, 2, 20, 21] applied to linear 2D parabolic equations with Bitsadze-Samarskii type NBC in one direction. The spectrum of matrix for this finite-difference problem is complicated [24]. Namely, eigenvalues of a finite difference problem can be positive and simple or positive and some of them multiple, or a few eigenvalues can be negative, or one of the eigenvalues is zero, or some eigenvalues are complex with positive or negative real parts. We show that the ADI method can be stable or nonstable for different values of parameters in NBC.

The paper is organized as follows. In Section 2 we present a differential problem and write the Peaceman-Rachford ADI method. In Section 3, we present some theoretical ADI method stability results for solution of a discrete problem. The results of numerical experiments are given in Section 4. The paper ends with some concluding remarks in Section 5.

## 2. The ADI Method for a 2D Parabolic Equation with a Nonlocal Boundary Condition in One Direction

### 2.1. Differential problem with a nonlocal condition

We consider a two dimensional parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \quad t \in (0,T), \tag{2.1}$$

in the rectangular domain $D = \{0 < x < L_x, 0 < y < L_y\}$ with boundary conditions

$$u(x,0,t) = w_l(x,t), \quad u(x,L_y,t) = w_r(x,t), \quad x \in [0,L_x], \tag{2.2}$$

$$u(0,y,t) = v_l(y,t), \quad y \in [0,L_y] \tag{2.3}$$

and additional Bitsadze-Samarskii type NBC with $0 < \xi < L_x$:

$$u(L_x,y,t) = \gamma u(\xi,y,t) + v_r(y,t), \quad y \in [0,L_y], \tag{2.4}$$

where $\gamma \in \mathbb{R}$, and the initial condition

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in \bar{D} = \{0 \leqslant x \leqslant L_x, 0 \leqslant y \leqslant L_y\}. \tag{2.5}$$

We are interested in sufficiently smooth solutions of this problem with NBC.

### 2.2. Notation

We introduce grids with uniform steps

$$\overline{\omega}_x^h := \{x_0 = 0,\ x_1,\ \ldots,\ x_n = L_x\}, \quad h_x = x_i - x_{i-1} = \frac{L_x}{n},$$

$$\overline{\omega}_y^h := \{y_0 = 0,\ y_1,\ \ldots,\ y_m = L_y\}, \quad h_y = y_j - y_{j-1} = \frac{L_y}{m},$$

$$\overline{\omega}^\tau := \{t_0 = 0,\ t_1,\ \ldots,\ t_N = T\}, \quad \tau = t_k - t_{k-1} = \frac{T}{N},$$

$\omega_x^h := \{x_1, \ldots, x_{n-1}\}$, $\omega_y^h := \{y_1, \ldots, y_{m-1}\}$, $\omega^\tau := \{t_1, \ldots, t_N\}$ and

$$\omega^\tau_{1/2} := \Big\{t_{1/2},\ \ldots,\ t_{N-1/2}\ :\ t_{k-1/2} = \frac{t_k + t_{k-1}}{2}\Big\}.$$

In the domain $\bar{D}$ we use grids $\overline{\omega} := \overline{\omega}^h_x \times \overline{\omega}^h_y$, $\overline{\omega}_y := \omega^h_x \times \overline{\omega}^h_y$, $\overline{\omega}_x := \overline{\omega}^h_x \times \omega^h_y$, $\omega := \omega^h_x \times \omega^h_y$ and $\partial\omega := \overline{\omega} \setminus \omega$. We also suppose that $\xi = sh_x$, $0 < s < n$. So, $(\xi, y_j) = (x_s, y_j)$ is a grid point.

Let $H$ be a space of grid functions $U_{i,j} := U(x_i, y_j)$ on $\omega$ with the inner product

$$(U, V) := \sum_{(x_i, x_j)\in\omega} U(x_i, y_j)V(x_i, y_j)\, h_x h_y.$$

We choose one of the most obvious orderings and set a vector

$$\mathbf{U} := [U_{11}, \ldots, U_{n-1,1}, U_{12}, \ldots, U_{n-1,m-1}] = (U_{11}, \ldots, U_{n-1,m-1})^T.$$

We use the notation $U^k_{i,j} := U(x_i, y_j, t_k)$ for functions defined on the grid (or parts of this grid) $\overline{\omega}\times\overline{\omega}^\tau$ and $U^{k-1/2}_{i,j} := U(x_i, y_j, t_{k-1/2})$ on the grid $\overline{\omega}\times\overline{\omega}^\tau_{1/2}$. We omit indices if they are the same in the whole equation. Let us define space grid operators

$$\delta^2_x U_{i,j} := \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2_x}, \quad \delta^2_y U_{i,j} := \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{h^2_y},$$

the time grid operator

$$\overline{\partial}_t U^k := \frac{U^k - U^{k-1}}{\tau}$$

and special grid functions

$$\widetilde{U}^{k-1/2}_x := \frac{U^k - 0.5\tau\delta^2_x U^k + U^{k-1} + 0.5\tau\delta^2_x U^{k-1}}{2}, \ (x_i, y_j) \in \overline{\omega}_y,$$

$$\widetilde{U}^{k-1/2}_y := \frac{U^k - 0.5\tau\delta^2_y U^k + U^{k-1} + 0.5\tau\delta^2_y U^{k-1}}{2}, \ (x_i, y_j) \in \overline{\omega}_x.$$

These functions will be used in the approximation of boundary conditions for a half-step of the ADI method. For simplification of writing letters with indices we denote $U := U^k$, $\overline{U} := U^{k-1/2}$, $\check{U} := U^{k-1}$.

### 2.3. Alternating Direction Implicit Method

We use the Peaceman–Rachford method for solving parabolic problem (2.1)–(2.5). The idea here is to alternate direction and thus solve two one-dimensional problems at each time step. We approximate the functions $f$, $v_l$, $v_r$, $w_l$, $w_r$, $u_0$ by $\overline{F}$, $V_l$, $V_r$, $W_l$, $W_r$, $U_0$.

In the first step we evaluate the derivative with respect to $y$ implicitly, the derivative with respect to $x$ explicitly, and use a time step of $0.5\tau$:

$$\frac{\overline{U} - \check{U}}{0.5\tau} = \delta^2_x \check{U} + \delta^2_y \overline{U} + \overline{F}, \quad (x_i, y_j) \in \omega, \tag{2.6}$$

with revised first type boundary conditions

$$\overline{U}|_{j=0} = \widetilde{W}_{lx}, \quad \overline{U}|_{j=m} = \widetilde{W}_{rx}, \quad x_i \in \omega_x^h. \tag{2.7}$$

In the second step, (now the approximation is explicit in $y$ and implicit in $x$) we have a finite difference scheme

$$\frac{U - \overline{U}}{0.5\tau} = \delta_x^2 U + \delta_y^2 \overline{U} + \overline{F}, \quad (x_i, y_j) \in \omega, \tag{2.8}$$

with simple first type NBC

$$U|_{i=0} = V_l, \quad U|_{i=n} = \gamma U|_{i=s} + V_r, \quad y_j \in \omega_y^h. \tag{2.9}$$

We approximate the initial condition as follows

$$U^0 = U_0, \quad (x_i, y_j) \in \overline{\omega}. \tag{2.10}$$

We also assume that all boundary and initial conditions are compatible. Note that $\overline{U} = \widetilde{U}_x$ for $(x_i, y_j) \in \omega$.

Let us introduce $(n-1) \times (n-1)$ and $(m-1) \times (m-1)$ matrices and $(n-1) \times 1$ vectors

$$\Lambda_x := \frac{1}{h_x^2} \begin{pmatrix} -2 & 1 & 0 \dots & 0 & 0 \\ 1 & -2 & 1 \dots & 0 & 0 \\ 0 & 1 & -2 \dots & 0 & 0 \\ \dots & \dots & \dots\dots & \dots & \dots \\ 0 & 0 & 0 \dots & -2 & 1 \\ 0 & 0 & \gamma \dots & 1 & -2 \end{pmatrix}, \quad \Lambda_y := \frac{1}{h_y^2} \begin{pmatrix} -2 & 1 & 0 \dots & 0 & 0 \\ 1 & -2 & 1 \dots & 0 & 0 \\ 0 & 1 & -2 \dots & 0 & 0 \\ \dots & \dots & \dots\dots & \dots & \dots \\ 0 & 0 & 0 \dots & -2 & 1 \\ 0 & 0 & 0 \dots & 1 & -2 \end{pmatrix},$$

$$\mathrm{V}_j := [V_{lj}, 0, \dots, 0, V_{rj}], \qquad \widetilde{\mathrm{W}}_l := [(\widetilde{W}_{lx})_1, (\widetilde{W}_{lx})_2, \dots, (\widetilde{W}_{lx})_{n-1}],$$

$$\widetilde{\mathrm{V}}_j := [(\widetilde{V}_{ly})_1, 0, \dots, 0, (\widetilde{V}_{ry})_j], \quad \widetilde{\mathrm{W}}_r := [(\widetilde{W}_{rx})_1, (\widetilde{W}_{rx})_2, \dots, (\widetilde{W}_{rx})_{n-1}],$$

$$\overline{\mathrm{F}}_j := [\overline{F}_{1j}, \overline{F}_{2j}, \dots, \overline{F}_{n-1,j}], \quad j = 1, \dots, m-1,$$

where $\gamma$ is in the $s$-th column of the matrix $\Lambda_x$. Let $\mathrm{I}_x$ be the identity matrix $(n-1) \times (n-1)$, $\mathrm{I}_y$ be the identity matrix $(m-1) \times (m-1)$ and $\mathbf{I} := I_y \otimes I_x$, where $A \otimes B$ denotes the Kronecker (tensor) product of matrices $A$ and $B$. Then we define $(m-1)(n-1) \times (m-1)(n-1)$ matrices and $(m-1)(n-1) \times 1$ vectors

$$\mathbf{A}_1 := -I_y \otimes \Lambda_x, \quad \mathbf{A}_2 := -\Lambda_y \otimes I_x, \quad \overline{\mathbf{F}} := [\overline{\mathrm{F}}_1, \overline{\mathrm{F}}_2, \dots, \overline{\mathrm{F}}_{m-1}],$$

$$\mathbf{V} := [\mathrm{V}_1, \mathrm{V}_2, \dots, \mathrm{V}_{m-1}], \quad \widetilde{\mathbf{W}} := [\widetilde{\mathrm{W}}_l, 0, \dots, 0, \widetilde{\mathrm{W}}_r],$$

$$\widetilde{\mathbf{V}} := [\widetilde{\mathrm{V}}_1, \widetilde{\mathrm{V}}_2, \dots, \widetilde{\mathrm{V}}_{m-1}].$$

We can directly verify that

$$\mathbf{A}_1 \mathbf{A}_2 = \mathbf{A}_2 \mathbf{A}_1 = \Lambda_y \otimes \Lambda_x. \tag{2.11}$$

We can consider FDS (2.6)–(2.10) as operator equations in the space $H$

$$(I + 0.5\tau A_2)\overline{U} = (I - 0.5\tau A_1)\check{U} + 0.5\tau \overline{F} + \frac{\tau}{2h_y^2}\widetilde{W} + \frac{\tau}{2h_x^2}\check{V}, \qquad (2.12)$$

$$(I + 0.5\tau A_1)U = (I - 0.5\tau A_2)\overline{U} + 0.5\tau \overline{F} + \frac{\tau}{2h_y^2}\widetilde{W} + \frac{\tau}{2h_x^2}V. \qquad (2.13)$$

The Laplace differential (or finite difference) operator can be split in a natural way as the sum of two operators. In our case, these operators are $A_1$ and $A_2$. Let us express $\overline{U}$ (see, system (2.12)–(2.13)) as follows

$$2\overline{U} = (I + 0.5\tau A_1)U + (I - 0.5\tau A_1)\check{U} - \frac{\tau}{2h_x^2}\big(V - \check{V}\big)$$

and let us eliminate $\overline{U}$ from this system. We get a two-level finite difference scheme

$$U = S\check{U} + \tau F$$

with the operator

$$S = (I + 0.5\tau A_1)^{-1}(I + 0.5\tau A_2)^{-1}(I - 0.5\tau A_2)(I - 0.5\tau A_1)$$

and

$$F = \overline{F} + \frac{\widetilde{W}}{h_y^2} + \frac{\widetilde{V}}{h_x^2} + \frac{\tau^2}{4h_y^2h_x^2}\Big((\overline{\partial}_t V_l)_0 + (\overline{\partial}_t V_r)_0 + (\overline{\partial}_t V_l)_m + (\overline{\partial}_t V_r)_m\Big).$$

In the first step of the Peacemen-Rachford method, we sweep in the $y$-direction and along each line $x_i = const$, $i = 1, \dots, n-1$ we have 1D subproblems (see, Eqs. (2.6)–(2.7)) with the tridiagonal matrix $\Lambda_y$. In the second step of the method, we sweep in the $x$-direction and along each line $y_j = const$, $j = 1, \dots, m-1$ we have 1D subproblems (see, Eqs. (2.8)–(2.9)) with the quasi-tridiagonal matrix $\Lambda_x$. When $\gamma = 0$, i.e. the classical boundary conditions are formulated, both matrices are symmetrical and tridiagonal. These tridiagonal systems can be solved very efficiently using Thomas' algorithm. We use a modification of this algorithm [22] for solving a quasi-tridiagonal system, for example, systems with matrix $\Lambda_x$.

In the commutative case (see, (2.11)), the Peaceman–Rachford method approximates the exact solution with the second order in time and space [21].

## 3. Investigation of Stability

The Peacemen-Rachford method is unconditionally stable in the case of the classical boundary conditions (the case $\gamma = 0$) [21]. Most stability theorems of this method require that both operators be symmetrical and positive, but, in the case of NBC, we have at least one non-selfconjugate operator. A spectrum of such operators is more complicated. So, additional investigation of stability properties is needed in this case.

We offer some theoretical results about the spectral properties of the matrices $\Lambda_x(\gamma, h_x, n)$ and $\Lambda_y(h_y, m)$. Let us consider the matrix

$$\Lambda = \Lambda(\gamma, h, n) := \Lambda_x(\gamma, h, n)$$

and suppose that $nh = L$. We formulate the main results about the spectrum for matrix $\Lambda$ and apply these results to the matrices $\Lambda_x = \Lambda(\gamma, h_x, n)$, $\Lambda_y = \Lambda(0, h_y, m)$.

Let us consider the domain $\Omega_q$ of the complex plane $\mathbb{C}_q$ consisting of the strip $0 < \operatorname{Re} q < \pi L/h$ and two rays $q = \beta\mathrm{i}$, $\beta > 0$ and $q = \pi L/h + \mathrm{i}\delta$, $\delta > 0$. The analytic function $\lambda = \frac{4}{h^2}\sin^2\frac{hq}{2L}$ is a one-to-one mapping of $\Omega_q$ onto $\Omega_\lambda \setminus \{0, 4/h^2\}$. Note that this mapping takes the points $q = 0$ and $q = \pi L/h$ into the points $\lambda = 0$ and $\lambda = 4/h^2$, respectively. We denote $\overline{\Omega}_q = \Omega_q \cup \{0, \pi L/h\}$. The eigenvectors and eigenvalues of the matrix $(-\Lambda)$ are of the form (see, [24]):

$$U_i(q) = \sin\frac{qx_i}{L}, \quad \lambda(q) = \frac{4}{h^2}\sin^2\frac{hq}{2L}, \quad \text{if } q \in \mathbb{C}_q \setminus \{0, \pi L/h\},$$
$$U_i(0) = \frac{x_i}{L}, \quad \lambda(0) = 0, \quad U_i(\pi L/h) = \frac{(-1)^{n-i}\,x_i}{L}, \quad \lambda(\pi L/h) = \frac{4}{h^2}.$$

For real $\lambda \neq 0$, $\lambda \neq \frac{4}{h^2}$ ($q \neq 0$, $q \neq \frac{\pi L}{h}$), we have

$$U_i(q) = \sin\frac{\alpha x_i}{L}, \qquad \lambda = \frac{4}{h^2}\sin^2\frac{h\alpha}{2L}, \qquad \text{if } q = \alpha \in \Bigl(0, \frac{\pi L}{h}\Bigr), \tag{3.1}$$
$$U_i(q) = \sinh\frac{\beta x_i}{L}, \qquad \lambda = -\frac{4}{h^2}\sinh^2\frac{h\beta}{2L}, \quad \text{if } q = \beta\mathrm{i},\ \beta > 0, \tag{3.2}$$
$$U_i(q) = (-1)^{n-i}\sinh\frac{\delta x_i}{L}, \quad \lambda = \frac{4}{h^2}\cosh^2\frac{h\delta}{2L}, \quad \text{if } q = \frac{\pi L}{h} + \mathrm{i}\delta, \delta > 0. \tag{3.3}$$

We find the values of $q \in \mathbb{C}_q \setminus \{0, \pi L/h\}$ as the roots of the equation

$$\sin q - \gamma\sin(q\tilde{\xi}) = 0, \tag{3.4}$$

and $q = 0$, if $\gamma = 1/\tilde{\xi}$, and $q = \pi L/h$, if $\gamma = (-1)^{n-s}/\tilde{\xi}$, where $\tilde{\xi} = s/n = \xi/L$.

For every $\gamma$ there exist $n-1$ such values $q_k$, $k = 1, \ldots, n-1$ in $\overline{\Omega}_q$, but they can be complex and multiple. A few results about the eigenvalues and eigenvectors for matrix $(-\Lambda)$ are proved in [24]).

**Lemma 1.** *For every $\tilde{\xi} \in (0; 1)$ there exist such $\gamma_- = \gamma_-(\tilde{\xi})$ and $\gamma_+ = \gamma_+(\tilde{\xi})$, $-1/\tilde{\xi} \leqslant \gamma_- \leqslant -1$, $1 \leqslant \gamma_+ \leqslant 1/\tilde{\xi}$, that for all $\gamma \in (\gamma_-; \gamma_+)$ all eigenvalues of the matrix $(-\Lambda)$ are positive and simple:*

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{h\alpha_k}{2L}, \quad \sin\alpha_k - \gamma\sin(\alpha_k\tilde{\xi}) = 0, \quad k = 1, 2, \ldots, n-1,$$

*and the corresponding eigenvectors*

$$U^{(k)} = \left[\sin\frac{\alpha_k x_1}{L},\ \ldots,\ \sin\frac{\alpha_k x_{n-1}}{L}\right],\quad k = 1, 2, \ldots, n-1,$$

*are linear independent, i.e., $(-\Lambda)$ is a matrix of a simple structure.*

*For $\gamma = \gamma_-$ or $\gamma = \gamma_+$ there exist positive multiple eigenvalues with one eigenvector. In this case $(-\Lambda)$ is not a matrix of simple structure. For $\gamma_+ = 1/\tilde{\xi}$, there exists a simple zero eigenvalue $\lambda_1(\gamma, \tilde{\xi})$, for $\gamma_+ > 1/\tilde{\xi}$ there exists only one simple negative eigenvalue $\lambda_1(\gamma, \tilde{\xi})$.*

*For every $\tilde{\xi} \in (0; 1)$, there exist $\gamma_-^* = \gamma_-^*(\tilde{\xi})$ and $\gamma_+^* = \gamma_+^*(\tilde{\xi})$, $\gamma_-^* < \gamma_-$, $\gamma_+ \leqslant \gamma_+^*$, such that for all $\gamma \in (\gamma_-^*; \gamma_+^*)$ and for all complex eigenvalues of the matrix $(-\Lambda)$ the inequality $\operatorname{Re}\lambda_k > 0$ is valid. The eigenvalues can be real or complex for such $\gamma$, and we have a full system of linear independent eigenvectors, except some $\gamma_1, \ldots, \gamma_{n_c} \in [\max\{\gamma_-^*, -1/\tilde{\xi}\}; \gamma_-] \cup [\gamma_+; \min\{\gamma_+^*, 1/\tilde{\xi}\}]$, $n_c(\tilde{\xi}) \geqslant 0$, where we have real multiple eigenvalues with an eigenvector and generalized eigenvectors of the first or second order.*

**Table 1.** The values of $\gamma_-$, $\gamma_+$, $\gamma_-^*$ and $\gamma_+^*$. $\mathbb{R}$ denotes the case, where there exist only real eigenvalues and $\mathbb{C}_+$ denotes real parts of all eigenvalues positive for all $\gamma$.

| | $\tilde{\xi}$ | 1/8 | 1/4 | 1/3 | 3/8 | 1/2 | 5/8 | 2/3 | 3/4 | 7/8 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\gamma_+^*$ | $n=2$ | – | – | – | – | $\mathbb{R}$ | – | – | – | – |
| | $n=3$ | – | – | $\mathbb{R}$ | – | – | – | $\mathbb{R}$ | – | – |
| | $n=4$ | – | $\mathbb{C}_+$ | – | – | $\mathbb{R}$ | – | – | $\mathbb{R}$ | – |
| | $n=6$ | – | – | $+\infty$ | – | $\mathbb{C}_+$ | – | $\mathbb{R}$ | – | – |
| | $n=8$ | 1794 | 2671 | – | 10034 | $+\infty$ | $\mathbb{C}_+$ | - | $\mathbb{R}$ | $\mathbb{R}$ |
| | $n=12$ | – | 943 | 1155 | – | 2702 | – | $+\infty$ | $\mathbb{C}_+$ | – |
| | $n=24$ | 651 | 599 | 634 | 649 | 729 | 959 | 1154 | 2702 | $\mathbb{C}_+$ |
| | $n=\infty$ | 589 | 524 | 536 | 536 | 535 | 535 | 535 | 535 | 535 |
| $\gamma_+$ | | 1.02 | 1.09 | 3 | 1.02 | 2 | 1.03 | 1.5 | 1.33 | 1.14 |
| $\gamma_-$ | | -1.02 | -1.09 | -1 | -1.02 | -2 | -1.03 | -1.5 | -1.33 | -1.14 |
| $\gamma_-^*$ | $n=2$ | – | – | – | – | $\mathbb{R}$ | – | – | – | – |
| | $n=3$ | – | – | $-\infty$ | – | – | – | $\mathbb{R}$ | – | – |
| | $n=4$ | – | -56 | – | – | $-\infty$ | – | – | $\mathbb{R}$ | – |
| | $n=6$ | – | – | -33 | – | -52 | – | $-\infty$ | – | – |
| | $n=8$ | -44.5 | -28 | – | -28.3 | -34 | -52 | – | $-\infty$ | $\mathbb{R}$ |
| | $n=12$ | – | -25.5 | -24.2 | – | -27 | – | -34 | -51.98 | – |
| | $n=24$ | -39.4 | -24.1 | -22.7 | -22.9 | -24 | 24.7 | -25.2 | -27 | -51.98 |
| | $n=\infty$ | -38.9 | -23.7 | -22.2 | -22.4 | -23.2 | -23.1 | -23.1 | -23.1 | -23.1 |

The values of $\gamma_-$, $\gamma_+$, $\gamma_-^*$ and $\gamma_+^*$ are presented in Table 1 for different values of parameters $\tilde{\xi}$ and $n = L_x/h_x$.

Let the eigenvalues of matrix $\mathbf{A}_{m\times m}$ be denoted by $\{\lambda_i\}_{i=1}^m$ and let the eigenvalues of matrix $\mathbf{B}_{n\times n}$ be denoted by $\{\mu_j\}_{j=1}^n$. We denote the corresponding eigenvectors as $\{u_i\}_{i=1}^m$ and $\{v_j\}_{j=1}^n$, respectively. Then the eigenvalues of

$\mathbf{A}\otimes\mathbf{I}_n+\mathbf{I}_m\otimes\mathbf{B}$ are the numbers $\{\lambda_i+\mu_j\}_{i=1}^{m}{}_{j=1}^{n}$, the eigenvalues of $\mathbf{A}\otimes\mathbf{B}$ are the numbers $\{\lambda_i\mu_j\}_{i=1}^{m}{}_{j=1}^{n}$ and $\{u_i\otimes v_j\}_{i=1}^{m}{}_{j=1}^{n}$ are eigenvectors for both matrices [25].

We apply this proposition to matrices $\mathbf{A}_1$ and $\mathbf{A}_2$. The matrix $(-\Lambda_y)$ is a matrix of the simple structure: eigenvalues and eigenvectors are given by

$$\mu^{(l)}=\frac{4}{h_y^2}\sin^2\frac{h_y\alpha_l}{2L_y},\quad V^{(l)}=\big[V_1^{(l)},\ \ldots,\ V_{m-1}^{(l)}\big],$$

where $V_j^{(l)}=\sin\dfrac{\alpha_l y_j}{L_y}$, $\alpha_l=\pi l,\ l=1,\ldots,m-1$. The matrix $(-\Lambda_x)$ is a matrix of more complicated structure, its eigenvalues and eigenvectors are given by

$$\lambda^{(k)}=\lambda^{(k)}(\gamma),\quad U^{(k)}=\big[U_1^{(k)}(\gamma),\ \ldots,\ U_{n-1}^{(k)}(\gamma)\big],\ \ k<n,$$

(see, Eqs. (3.1)–(3.4) and Lemma 1). If all the eigenvalues of the matrix $(-\Lambda_x)$ are simple (without multiple eigenvalues), then there exist $n-1$ linear independent eigenvectors, or else the number of linear independent eigenvectors is less than $n-1$.

**Lemma 2.** *If matrix $\mathbf{A}_1$ is a matrix of simple structure then matrices $\mathbf{A}_1\mathbf{A}_2$, $\mathbf{A}_2\mathbf{A}_1$ and $\mathbf{A}=\mathbf{A}_1+\mathbf{A}_2$ are matrices of simple structure and have the same system of eigenvectors:*

$$U^{(k,l)}=\{U_{ij}^{(k,l)}\},\ U_{ij}^{(k,l)}=U_i^{(k)}V_j^{(l)},\ \ k=1,\ldots,n-1,\ \ l=1,\ldots,m-1,$$

*and*

$$\lambda^{(k,l)}(\mathbf{A}_1\mathbf{A}_2)=\lambda^{(k,l)}(\mathbf{A}_2\mathbf{A}_1)=\lambda^{(k)}(\gamma)\mu^{(l)},$$
$$\lambda^{(k,l)}(\mathbf{A}_1+\mathbf{A}_2)=\lambda^{(k)}(\gamma)+\mu^{(l)},\ \ k=1,\ldots,n-1,\ \ l=1,\ldots,m-1.$$

*Corollary 1.* If $\mathbf{A}_1$ is a matrix of simple structure, then $\mathbf{S}$ is a matrix of simple structure too, and

$$\lambda(\mathbf{S})=\frac{\big(1-0.5\tau\lambda(\mathbf{A}_1)\big)\big(1-0.5\tau\lambda\big(\mathbf{A}_2\big)\big)}{\big(1+0.5\tau\lambda(\mathbf{A}_1)\big)\big(1+0.5\tau\lambda\big(\mathbf{A}_2\big)\big)}.\tag{3.5}$$

**Lemma 3.** *If matrix $\mathbf{A}_1$ is a matrix of simple structure and* $\operatorname{Re}\lambda^{(k)}>0$ *for all $k=1,\ldots,n-1$, then $|\lambda(\mathbf{S})|<1$.*

## 4. Numerical Experiments

In this section, we present the results of computational experiments. The aim of experiments is to test the usage of the ADI method for solving a parabolic problem with NBC.

Using the ADI method described above, three parabolic problems with NBC were tested. The behaviour of solutions $u(x,y,t)$ is different in these three examples as $t \to \infty$. We choose functions $f(x,y,t)$, $w_l(x,t)$, $w_r(x,t)$, $v_l(y,t)$, $v_r(y,t)$ and $u_0(x,y)$ so that these three differential problems have the solutions:

$$1)\ u_1 = x^3 + y^3 + t^3; \qquad 2)\ u_2 = \mathrm{e}^{x+y+t}; \qquad 3)\ u_3 = 2xy\sin t,$$

respectively. Each problem was solved with the following parameters

$$\xi_1 = 1/4,\ \ \xi_2 = 1/2,\ \ \xi_3 = 3/4,\ \ L_x = L_y = 1$$

(so, $\tilde{\xi} = \xi$). Test problems were solved with different values of parameters $\gamma$, $\tau$ and $h$. Numerical experiment results are presented for $\tau = 10^{-4}$, $h = 0.0125$ and $T = 2$ (Problems 1 and 2), and $T = 13$ (Problem 3).

**Table 2.** The intervals $[\gamma_m; \gamma_M]$.

| | Problem | 1 | 2 | 3 |
|---|---|---|---|---|
| $\gamma_m$ | $\xi = 1/4$ | -9.4 | -25. | -57.6 |
| | $\xi = 1/2$ | -29.0 | -34.1 | -35.9 |
| | $\xi = 3/4$ | -26.9 | -26.0 | -27.6 |

| | Problem | 1 | 2 | 3 |
|---|---|---|---|---|
| $\gamma_M$ | $\xi = 1/4$ | 4.0 | 6.0 | 11.7 |
| | $\xi = 1/2$ | 2.4 | 3.1 | 4.6 |
| | $\xi = 3/4$ | 1.6 | 1.8 | 2.1 |

**Table 3.** The values of the solutions and computational errors.

| Problem | 1 | 2 | 3 |
|---|---|---|---|
| $U(1;1/2;2)$ | 0.91252E+1 | 0.33116E+2 | |
| $U(1;1/2;13)$ | | | 0.42016E+0 |
| $\varepsilon$ | 0.600081E-3 | 0.26970E-2 | 0.89765E-4 |

The minimum $\gamma_m$ and the maximum $\gamma_M$ of the stability interval $[\gamma_m, \gamma_M]$ for the ADI method are presented in Table 2. The stability interval was established using a simple rule: the error of solution must be the same as that for the problem without NBC ($\gamma = 0$). Note that the ADI method is stable for all the three test problems in the case of the negative eigenvalue of matrix $\mathbf{A}_1$ where this eigenvalue is almost zero ($\gamma \gtrapprox 1/\xi$). This effect may be explained in part by the formula $\lambda(\mathbf{S}) = \tilde{\lambda}_1\tilde{\lambda}_2$, where

$$\tilde{\lambda}_i = \frac{1 - 0.5\tau\lambda(\mathbf{A}_i)}{1 + 0.5\tau\lambda(\mathbf{A}_i)}, \qquad i = 1, 2.$$

If at least one eigenvalue of matrix $\mathbf{A}_1$ is negative, then $|\tilde{\lambda}_1| > 1$, but $|\tilde{\lambda}_2| < 1$, so it may be $|\lambda(\mathbf{S})| < 1$. The values of $\gamma_m$ and $\gamma_M$ (Table 2) are different for the same $\xi$. This effect can not be explained by matrix $\mathbf{S}$ eigenvalues, because these eigenvalues don't depend on $f$, $v_l$, $v_r$, $w_l$, $w_r$ and $u_0$ expressions. So, additional investigation is needed to explain the stability conditions of the ADI method. An approximate value of solution $U$ and computational error $\varepsilon = \max_{ij} |U_{ij} - u_{ij}|$ are presented in Table 3. We have the greatest computational error at the point $(1; 1/2)$ in all the three problems. For $\gamma < \gamma_m$ or $\gamma > \gamma_M$, errors are growing. A typical error growing is shown in Table 4.

**Table 4.** Computational error.

$\xi = 1/4$ Problem 1

| $\gamma$ | $\varepsilon$ |
|---|---|
| -75.0 | 0.37050E+1 |
| -30.0 | 0.14887E-2 |
| -10.0 | 0.92220E-3 |
| -9.5 | 0.90408E-3 |
| -5.0 | 0.60081E-3 |
| 1.0 | 0.60081E-3 |
| 2.5 | 0.60081E-3 |
| 4.0 | 0.90122E-3 |
| 4.3 | 0.10014E-2 |
| 13.0 | 0.13848E-1 |
| 20.0 | 0.25756E+1 |

$\xi = 1/4$ Problem 2

| $\gamma$ | $\varepsilon$ |
|---|---|
| -60.0 | 0.15376E+0 |
| -40.0 | 0.50163E-2 |
| -20.0 | 0.36163E-2 |
| -11.0 | 0.26970E-2 |
| 4.5 | 0.26970E-2 |
| 4.6 | 0.27428E-2 |
| 6.0 | 0.40054E-2 |
| 10.0 | 0.11425E-1 |
| 15.0 | 0.14919E+0 |

$\xi = 1/2$ Problem 3

| $\gamma$ | $\varepsilon$ |
|---|---|
| -37.0 | 0.11745E+5 |
| -36.0. | 0.14627E-1 |
| -35.0 | 0.89765E-4 |
| 1.0 | 0.89765E-4 |
| 4.5 | 0.89765E-4 |
| 4.6 | 0.13483E-3 |
| 5.0 | 0.23469E-3 |
| 5.5 | 0.62273E+4 |

$\xi = 3/4$ Problem 2

| $\gamma$ | $\varepsilon$ |
|---|---|
| -28.0 | 0.12772E+4 |
| -26.1 | 0.48370E-2 |
| -26.0 | 0.31166E-2 |
| -10.0 | 0.26970E-2 |
| 1.0 | 0.26970E-2 |
| 1.6 | 0.26970E-2 |
| 1.7 | 0.31700E-2 |
| 2.3 | 0.63396E-1 |
| 2.6 | 0.13651E+2 |

## 5. Conclusions and Remarks

The values of parameters $\xi$ and $\gamma$ in NBC are essential for the stability of the ADI method. The sufficient stability condition $|\lambda(\mathbf{s})| < 1$ of this method is theoretically justified. The numerical experiment shows the efficiency of this theoretical condition. If $\xi$ is fixed, then the ADI method is stable for a sufficiently wide interval of $\gamma$ values, but this interval is not symmetrical with respect to $\gamma = 0$. It is determined by a FDS spectrum structure. Some numerical experiments are given and they validate the theoretical results.

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