Mathematical Modelling and Analysis
Volume 13 Number 2, 2008, pages 263–274
 ISSN 1392-6292 print, ISSN 1648-3510 online

NUMERICAL ANALYSIS OF LASING DYNAMICS IN VOLUME FREE ELECTRON LASER

S. SYTOVA

Research Institute for Nuclear Problems, Belarusian State University

Bobruiskaya 11, 220030 Minsk, Belarus

E-mail: `sytova@inp.minsk.by`

Received October 1, 2007; revised February 28, 2008; published online May 1, 2008

Abstract. Nonlinear phenomena originating in volume free electron laser (VFEL) are investigated by methods of mathematical modelling using computer code VOLC. It was demonstrated the possibility of excitation of quasiperiodic oscillations not far from threshold values of electron beam current density and VFEL resonator length. It was investigated sensibility of numerical solution to initial conditions for different VFEL regimes of operation. Parametric maps with respect to electron beam current and detuning from synchronism condition present complicated root to chaos with windows of periodicity in VFEL. Investigation of chaotic lasing dynamics in VFEL is important in the light of experimental development of VFEL in Research Institute for Nuclear Problems.

Key words: free electron laser, simulation, nonlinear integro-differential system, dynamical system, chaos.

1 Introduction

Chaotic dynamics means the tendency of a wide range of systems to transition between different states with deterministic periodic and non-periodic behavior. Under modern concept of deterministic chaos [29] dynamical system under changing of external control parameters gives series of bifurcations leading to meshing of self-oscillations right up to stochastic oscillations with continuous spectrum. Examples of such nonlinear physical systems are nonlinear optical devices, lasers, particle accelerators, free electron lasers (FELs) etc.

Nonlinearity is necessary but non-sufficient condition for chaos in the system. The main origin of chaos is the exponential divergence of initially close trajectories in the nonlinear systems. This is so-called the Butterfly effect [25]. The heart of the problem is the sensibility of the system to initial condition.

Different types of FELs are the main object for analysis of its chaotic nature in this paper. FELs are tunable high-power sources of coherent radiation. This radiation can be obtained in different wavelength ranges and it is the result of transformation of kinetic energy of a high energy electron beam. FELs go back to Madey [26] who investigated stimulated emission of bremsstrahlung by a relativistic electron into a single electromagnetic mode of a parallel light beam. Electron and light beam were considered moving through a periodic transverse magnetic field. In 1976 a group of Stanford University under the direction of Madey demonstrated first lasing of the FEL first prototype at a wavelength of 3.4 mm [12].

The main principle of electronic devices such as FEL [28], [21] the same as traveling wave tubes (TWT), backward wave tubes (BWT), backward wave oscillators (BWO) and other vacuum electronic devices [19] is based on radiation of bunches of charged particles which are under conditions of synchronism moving uniformly in the slow-wave structure (resonator or undulaotr). Distributed feedback provides coupling between electromagnetic field and electron beam.

Interaction of electron beam and electromagnetic field under distributed feedback is the main origin of self-oscillations in vacuum electronic devices [18, 23, 20]. In [8, 11] theoretical and experimental investigations of the super-ACO FEL have been made to observe clear bifurcation and chaos sequences in the response of the FEL to a detuning modulation that is a changing of the synchronism between the electron beam and the optical pulse. In [24] different regimes of so-called "weak" chaos and "hyperchaos" or self-oscillations were investigated for BWT. In [23] it was investigated a nonlinear dynamics of BWT in the presence of energy dissipation at wave transmission, field of space charge, wave reflection at system edges. It was depicted main principles of chaos control in BWT via suppression of self-modulation. In [13] methods based on the reshaping the inner topology of the single-particle phase-space to stabilize the oscillations of the FEL intensity in the deep saturated regime are proposed. In [9] the chaotic sea model for FEL dynamics is considered for investigation of phase space portraits of evolution of FEL radiation intensity at different times. Three main routes to chaos for nonlinear systems such as FEL and driven plasma diodes under changes of control parameters were investigated [20, 22]. There are period doubling, quasiperiodicity and intermittency. So, investigation of chaos in electronic devices is of great interest in modern physics.

One of possible types of FELs is volume free electron laser (VFEL). First lasing of VFEL in mm wavelength range obtained recently [14] put the beginning of experimental developing of new type of electronic generators. Their functioning is based on principles of multi-wave volume (non-one-dimensional) distributed feedback (VDFB) where electromagnetic waves and electron beam spread angularly one to the other [3]. Usually in FEL, TWT, BWT the distributed feedback is one-dimensional when electron beam and electromagnetic waves move in one line. In VFEL operation the linear stage investigated analytically [2] quickly changes into the nonlinear one where most of the electron beam energy is transformed into electromagnetic radiation. In [1] it is emphasized that VFELs are one of several attractive alternatives to ordinary FELs, because they are more compact devices capable to operate from submillimeter to X-ray wavelength ranges. We faced the challenge of oscillations in VFEL simulation. In VFEL chaotic dynamics is induced by interaction of electron beam and electromagnetic field under VDFB.

Mathematical model and numerical methods for VFEL nonlinear stage simulation were proposed in [4, 5]. They are implemented in computer code VOLC [6]. Different VFEL geometries were investigated numerically in [7, 30]. All numerical results are in good agreement with analytical predictions. Experiments on VFEL go on [15] and need optimal geometry determination and result processing.

So, investigation of chaos in VFEL is important in the light of its experimental development. As more than ten control parameters are in the system it is very complicated to investigate the full picture of possible chaotic behaviour. In [6, 7, 30] a gallery of different chaotic regimes for VFEL laser intensity with corresponding attractors and Poincaré maps was proposed. There are periodic, quasiperiodic regimes and chaotic self-oscillations. Solution bifurcation points corresponding to transitions between different regimes of generation were investigated.

In this paper new attempts were undertaken to investigate numerically the root reason to chaos in VFEL. This is necessary to detect and validate some new and complex phenomena that cannot be examined through analytical investigation.

2 VFEL Mathematical Formulation

Functioning of experimental installations [14] and [15] can be reduced to the following theoretical scheme of VFEL. It was proposed firstly in [4]. So, let us assume that a relativistic electron beam with electron velocity $\mathbf{u}$ passes through a spatially periodic resonator of the length L. Under diffraction conditions some strong electromagnetic waves can be excited in the resonator. If simultaneously electrons of the beam are under synchronism condition, they emit electromagnetic radiation in directions depending on diffraction conditions. System of equations describing VFEL is obtained from Maxwell equations in the slowly varying envelope approximation under following assumptions. The initial electromagnetic field is uniform over the entire electron beam or can follow to Gaussian distribution or can be equal to zero. In the last case we deal with regime of oscillator of VFEL. The resonator is quite large in transverse directions so the slippage effects at the edges of the system are ignored. Electron charge density is uniform or can follow to Gaussian distribution too.

For two-wave VFEL the system of equations have the following form:

$$\begin{aligned}
&\frac{\partial E}{\partial t} + a_1 \frac{\partial E}{\partial z} + b_{11} E + b_{12} E_\tau = \Phi \int_0^{2\pi} \frac{2\pi - p}{8\pi^2} \left(\exp(-i\Theta(t,z,p)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. + \exp(-i\Theta(t,z,-p)\right)\, dp, \\
&\frac{\partial E_\tau}{\partial t} + a_2 \frac{\partial E_\tau}{\partial z} + b_{21} E + b_{22} E_\tau = 0,
\end{aligned} \tag{2.1}$$

$$\frac{d^2\Theta(t,z,p)}{dz^2} = \Psi\left(k - \frac{d\Theta(t,z,p)}{dz}\right)^3 \mathrm{Re}\left(E(t-z/u,z)\exp(i\Theta(t,z,p))\right),$$

$$E|_{z=0} = E_0, \quad E_\tau|_{z=L} = E_1,$$

$$E|_{t=0} = 0, \quad E_\tau|_{t=0} = 0, \quad \Theta(t,0,p) = p, \quad \frac{d\Theta(t,0,p)}{dz} = k - \omega/u,$$

where $t > 0$, $z \in [0, L]$, $p \in [-2\pi, 2\pi]$. Amplitudes of electromagnetic fields are denoted as $E(t,z)$, $E_\tau(t,z)$, function $\Theta(t,z,p)$ describes phase of electrons of the beam relative to electromagnetic wave, k is a projection of wave vector on axis z, ω is a field frequency. We suppose that all functions are smooth, bounded and slowly changing.

Three-wave VFEL was considered in [6], where principles of multi-wave VFEL simulation were proposed. In [18, 23, 24] and other papers the following system of equations is used to simulate different vacuum electronic devices:

$$\frac{\partial F}{\partial \tau} - \frac{\partial F}{\partial \zeta} = -\frac{1}{\pi}\int_0^{2\pi} \exp(-i\theta)d\theta_0, \tag{2.2}$$

$$\frac{\partial^2\theta}{\partial\zeta^2} = -\mathrm{Re}\left(F\exp(i\theta)\right),$$

$$F|_{\zeta=L} = 0, \quad \theta|_{\zeta=0} = \theta_0, \quad \frac{\partial\theta}{\partial\zeta}|_{\zeta=0} = 0.$$

System (2.2) is versatile in the sense that it remains the same within some normalization for a wide range of electronic devices (FEL, BWT, TWT etc). In (2.2) dynamics of electron beam is determined by time t_0 of electron incoming in the interaction zone and by corresponding phase $\theta_0 = \omega t_0$.

We take into account an extra spatially transverse coordinate of electron incoming in the interaction zone. Then the right-hand side of the first equation of system (2.1) is obtained by averaging over these two phases. So, this allows to simulate electron beam dynamics more precisely. Such consideration is very important when electron beam moves angularly to electromagnetic waves.

It is clear that the full three-dimensional model realized in computer code could allow to obtain ideal agreement between experimental and numerical data. But simplified 1D or 2D models like [10] or ours taking into account the principle physical mechanisms are also very efficient.

3 Numerical Tool for VFEL Simulation

There exists a wide range of FEL experimental setups functioning of which is simulated by multiple computer codes [16, 17, 27] and others. Along with the particle-in-cell method, a slowly varying envelope approximation [16] is widely used in FEL simulation and in computer codes such as MEDUSA [17] or GENESIS [27].

We also use this approximation in VFEL simulation (see [4] and our previous works cited there). Numerical methods for all possible two-wave and three-wave VFEL geometries including external reflectors proposed in previous works [4, 5] are implemented in computer code VOLC, version 1.0 [6]. VOLC means "VOLume Code". It was developed on the basis of multiple Fortran codes, created in 1991—2005 years. Dimensionality of the model is 2D (one spatial coordinate and one phase space coordinate) plus time. Three-wave geometries were considered to confirm all main VFEL physical laws and mechanisms. A good agreement between results of simulation and experimental data was obtained in many cases [15].

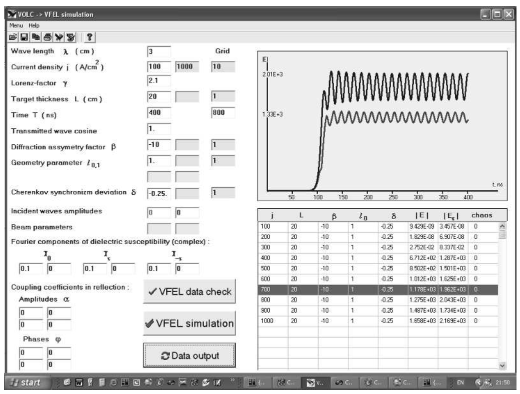

Figure 1. Interface of computer code VOLC.

New version 2.0 of VOLC allows for two-wave VFEL geometries to obtain distributions of VFEL intensities with respect to electron beam current density j, resonator length L, diffraction asymmetry factor, different system parameters, detuning from synchronism condition δ, as well as dynamical regimes recognition and intensity Fourier transforms. Reduction of VOLC possibilities to two-wave case is connected with geometry of experimental installation [15] for which VOLC simulation was designed. Interface of VOLC, version 2.0 is presented in Fig.1. It is a stand alone program written in Borland C++ Builder for use in the Windows Operating environment. The given interface allows to define input parameters, check their validity with corresponding messages, call the main routine for VFEL simulation and it also presents the output of some results in the window including 2D plots and a summary table of results. The main routine for VFEL simulation was elaborated in Compaq Visual Fortran and can operate without VOLC interface. In this case only the file with input parameters should be filled. Numerical results are written in specified files.

VOLC interface uses the standard MS Windows dialog and are supplied by screen tips. All wrong user actions are stopped with corresponding messages. Computation of distributions of intensities with respect to some parameters can take much of computer time, so at any moment user can stop computation without losses of calculated data.

4 Numerical Investigation of Chaos

Analytical investigation of chaos in the system (2.1) seems to be impossible because of its strong nonlinearity. There exists a wide range of external control parameters. Electron beam moving through resonator in VFEL leads to a diversity of features of generation dynamics that is due to non-local nature of interaction between electron beam and electromagnetic field under VDFB. Here the chaotic behavior in two-wave VFEL was investigated for the following set of parameters: wavelength $\lambda = 3$ cm, $L = 10 \div 40$ cm, $j = 400 \div 3000$ A/cm^2, $\delta kL = -20 \div 20$.

In VFEL theory there exist some threshold points that are bifurcation points. First threshold point corresponds to beginning of electron beam instability. Here regenerative amplification starts while the radiation gain of generating mode is less than radiation losses. Parameters at which radiation gain becomes equal to absorption correspond to the second threshold point after that generation progresses actively. Numerical results demonstrating transition between these points were obtained in [4]. Such threshold points exist with respect to the length of the resonator L too.

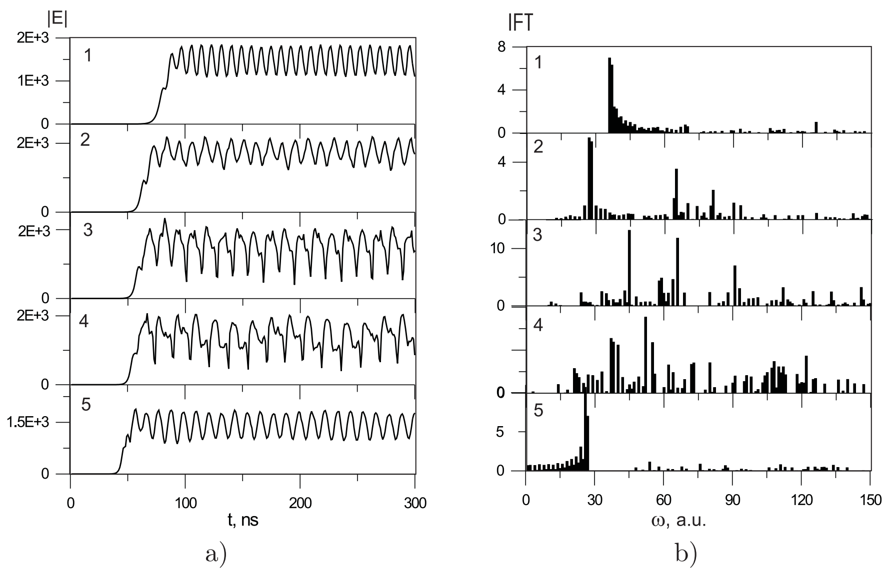

Figure 2. Transition between different regimes of VFEL intensity (a) and corresponding phase space portrait (b) for resonator length L equal to (1) 16 cm, (2) 18 cm, (3) 19 cm, (4) 21 cm and (5) 23 cm for $j = 2000$ A/cm^2.

In simulations an important VFEL feature due to VDFB was shown. This is the initiation of quasiperiodic regimes at relatively small values near threshold points with respect to current density and length of the resonator. In Fig. 2 the possibility of such initiation at relatively small resonator length is demonstrated. The threshold value is equal in this case to 13 cm. In the figure the transition from periodic to quasiperiodic regime, than through chaotic self-oscillations to periodic regime again is presented.

Unfortunately, it seems to be impossible to investigate analytically the stability and convergence of the solution of the difference system to the solution of the initial integro-differential problem because of strong nonlinearity of the given problem. So, the principal question in VFEL simulation just as in any simulations is the following. Either obtained oscillations are due to real chaotic nature of the system either they are induced by some miscalculations or finite order of approximation of difference system. One of the possible answers is to test the sensibility to initial conditions of numerical solutions because the main origin of chaos is the exponential divergence of initially close trajectories in the nonlinear system. If we obtain instability with respect to initial conditions for chaotic regimes and stability for periodic ones, this will be a good verification of numerical algorithms proposed. Let us consider numerical results in Fig.3.

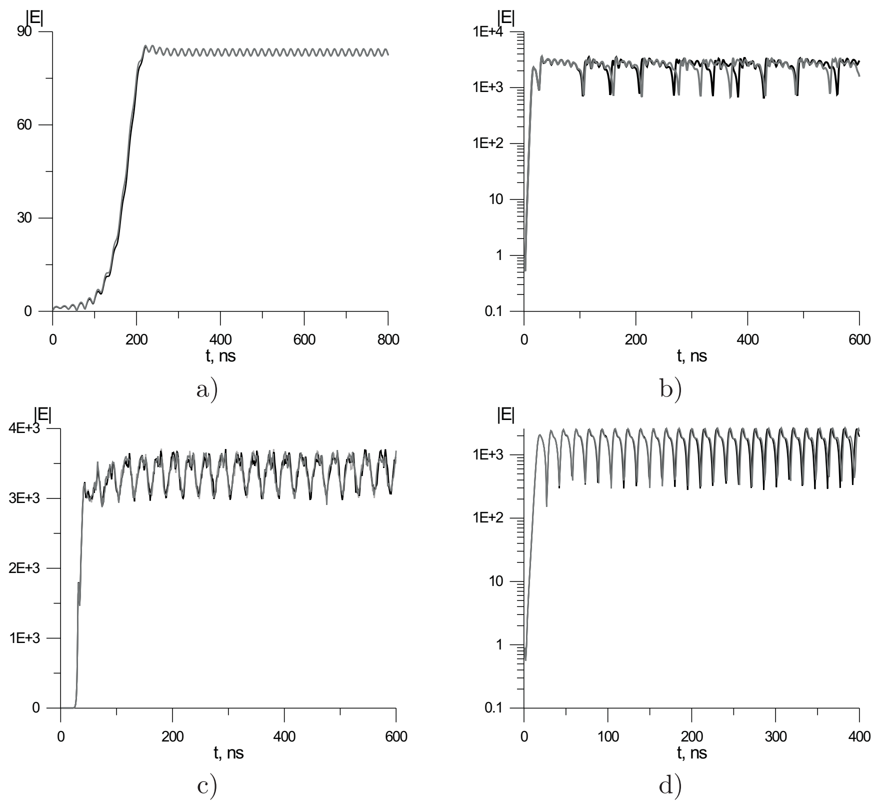

Figure 3. Sensibility to initial conditions for different VFEL regimes: (a) periodic, (b) and (c) chaotic, (d) quasiperiodic.

In each plot there are two curves (black and grey) corresponding to the following sets of parameters: (a) $j = 500$ A/cm^2, $|E_0| = 1$ and $|E_0| = 1.1$; (b) $j = 2300$ A/cm^2, $|E_0| = 1$ and $|E_0| = 1 + 10^{-15}$, (c) $j = 2000$ A/cm^2 and $j = 2000+10^{-8}$ A/cm^2, (d) $j = 1950$ A/cm^2 and $j = 1960$ A/cm^2. It is obvious that for periodic and quasiperiodic regimes (Fig.3a and Fig.3d) deviation up to ten percents doesn't lead to divergence of numerical results while for a chaotic case calculations with double precision and a change of the last digit in the value of current density leads to the Butterfly effect (Fig.3b).

Parametric maps demonstrating root to chaotic lasing in VFEL were obtained numerically with respect to electron beam current and detuning from synchronism condition δ expressed in relative units δkL for $L = 20$ cm . They are depicted in Fig.4 and Fig.5.

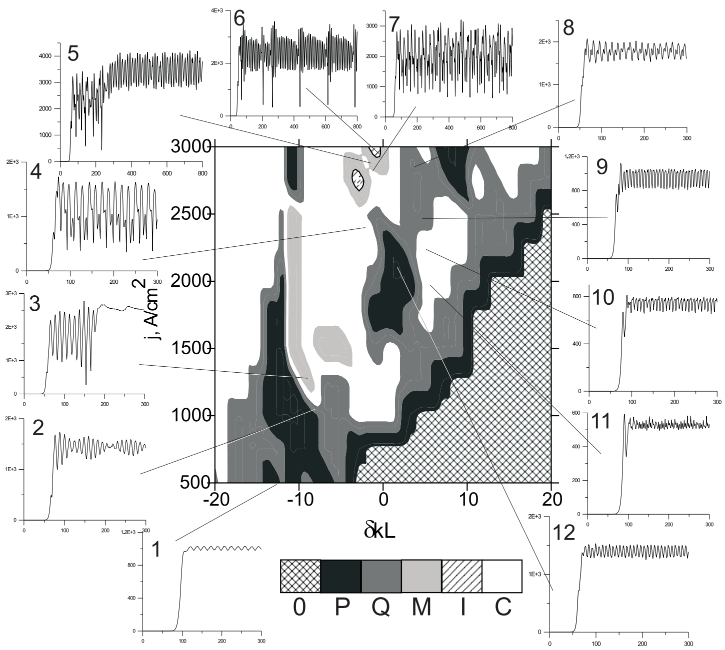

Figure 4. Two-parametric maps of chaotic lasing for transmitted wave. 0 depicts a domain under beam current threshold. P, Q, C correspond to periodic regimes, quasiperiodicity and chaos, respectively. M describes domains with transitions between large-scale and small-scale amplitudes. I stands for intermittency.

On edges of each chart the most typical dependencies of fields intensities on time (in ns) are presented. 0 depicts a domain under generation threshold where generation of electromagnetic radiation is not realized. After overcoming this threshold by parameters j and δ the radiation gain of generating mode becomes equal to absorption and generation begins and develops actively. All main dependencies of different threshold points were investigated analytically in [2] and numerically in [4, 7].

After small overcoming of this threshold the periodical self-oscillations are developed. This is illustrated in Fig.4 via black strip separating threshold zone from the remaining domain. Then the initiation of quasiperiodic regimes for transmitted wave is possible in VFEL. This is demonstrated in Fig.4 via grey strip near black one. Then transitions from periodic to quasiperiodic regimes and through non-periodical chaotic self-oscillations with some lines in spec-

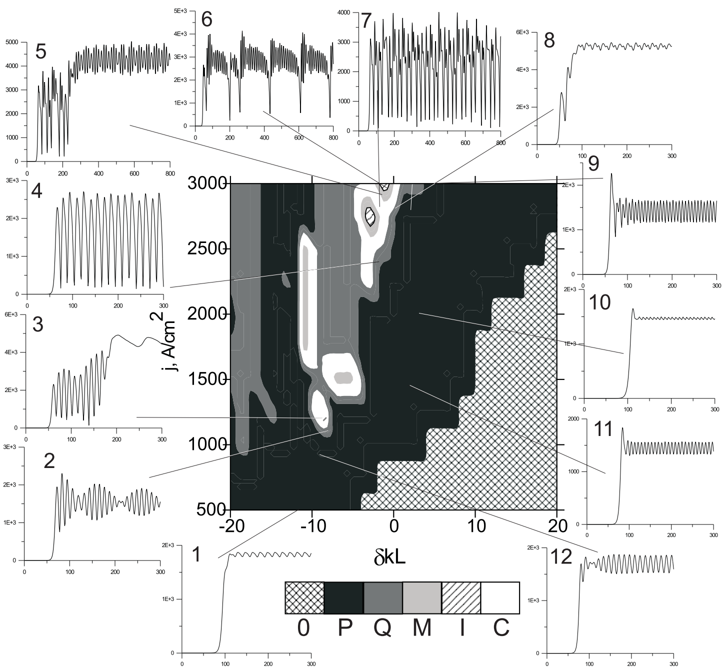

Figure 5. Two-parametric maps of chaotic lasing for diffracted wave. Notation is the same as in Fig.4.

trum to periodic regimes again are founded. For diffracted wave quasiperiodic regimes are realized at relatively large values of parameters.

Plots 1 and 12 in Fig.4 and plots 1 and 9–12 in Fig.5 demonstrate different periodic regimes realizing far from threshold. Plots 2, 8 and 9 in Fig.4 and 2, 4, 8 in Fig.5 depict examples of quasiperiodic regimes. Plots 2, 4, 8 are results of traditional Hopf bifurcation introducing new incommensurate frequency in the spectrum. Plot 9 is developed in consequence of disappearance of large number of frequencies from continuous chaotic spectrum. Plots 6 demonstrate intermittency. Plots 4, 7, 10, 11 (Fig.4) the same as plot 7 (Fig.5) are examples of different chaotic regimes.

In [24] BWT with strong reflections parametric maps with large-scale and small-scale amplitude regimes were adduced. In VFEL modelling besides domains with simply large-scale and small-scale amplitudes (compare plots 1 and 4 in both figures) we obtained domains with transitions between them (see plots 3, 5 in Fig.4-5). After such transition regimes with periodicity, quasiperiodicity and chaos can be established. Last type of transition is realized as a result of tangent bifurcation from chaotic attractor to another one. In the case of transition from large-scale to small-scale amplitudes regime when quasiperiodic regime is obtained, we deal with the tangent bifurcation from chaotic attractor to quasiperiodic movement on the torus. All this can be explained by the nonlinear mode competition mechanism in the system.

It follows from the presented simulations that the domain with small detuning δ from synchronism condition $|\omega - \mathbf{ku}| \leq \delta\omega$ and quite large values of current density j is most rich for transitions between different regimes.

5 Conclusions

Mathematical models and computer code VOLC described here can be used effectively in modelling of nonlinear regimes of VFEL operation.

In particular, it was demonstrated the possibility of excitation of quasiperiodic oscillations not far from threshold values of electron beam current density and resonator length. It was investigated sensibility of numerical solution to initial conditions for different regimes of VFEL operation that was a good verification of numerical algorithms. Parametric maps with respect to electron beam current and detuning from synchronism condition present complicated root to chaos in VFEL with windows of periodicity and quasiperiodicity.

Investigation of chaotic dynamics showed the possibility to choose more precisely domains with periodic self-modulation instead of chaotic one. They will be useful for providing experiments on VFEL on the installations created at the Research Institute for Nuclear Problems of Belarusian State University.

Acknowledgment

Author thanks Prof. V. G. Baryshevsky for permanent interest to her work.

References

[1] S. Alam, M.O. Rahman, C. Bentley and M. Ando. Basics of surface and volume FEL's). In *Proc. of the Second Asian Particle Accelerator Conference, Beijing, China, 2001*, pp. 277–280, 2001.

[2] V. Baryshevsky, K. Batrakov and I. Dubovskaya. Parametric (quasi-Cherenkov) X-ray FEL. *Journ. Phys. D*, **24**:1250–1257, 1991.

[3] V. Baryshevsky and I. Feranchuk. Parametric beam instability of relativistic charged particles in a crystal. *Phys. Let. A*, **102**:141–144, 1984.

[4] K. Batrakov and S. Sytova. Modelling of Volume Free Electron Lasers. *Computational Mathematics and Mathematical Physics*, **45**(4):666–676, 2005.

[5] K. Batrakov and S. Sytova. Nonstationary stage of quasi-Cherenkov beam instability in periodical structure. *Mathematical Modelling and Analysis*, **10**(1):1–8, 2005.

[6] K. Batrakov and S. Sytova. Mathematical modeling of multiwave Volume Free Electron Laser: basic principles and numerical experiments. *Mathematical Modelling and Analysis*, **11**(1):13–22, 2006.

[7] K. Batrakov and S. Sytova. Numerical simulation of nonlinear effects in Volume Free Electron Laser (VFEL). In *Proc. of the Russian Particle Accelerator Conference RUPAC2006, Novosibirsk, Russia, 2006*, pp. 141–143, 2006.

[8] M.E. Couprie. Chaos studies of the super-ACO free electron laser. *Nucl. Instr. Meth.*, **A507**:1–7, 2003.

[9] A. Antoniazzi et al. Wave-particle interaction: from plasma physics to the free-electron laser. *J. Phys.: Conf. Ser.*, **7**:143–153, 2005.

[10] A.V. Elzhov et al. 1.5-dimensional simulation of free electron maser amplifiers. *Phys. Rev. ST Accel. Beams*, **8**:090701, 2005.

[11] C. Bruni et al. Chaotic nature of the super-ACO FEL. *Nucl. Instr. Meth.*, **A528**:273–277, 2004.

[12] D.A.G. Deacon et al. First operation of a free-electron laser. *Phys. Rev. Let.*, **38**:892–894, 1977.

[13] R. Bachelard et al. Control of the intensity of a wave interacting with charged particles. In *Proc. of the 28th Intern. Free Electron Laser Conference FEL2006, Berlin, Germany, 2006*, pp. 83–86, 2006.

[14] V. Baryshevsky et al. First lasing of a volume FEL (VFEL) at a wavelength range 4-6 mm. *Nucl. Instr. Meth.*, **A483**:21–24, 2002.

[15] V. Baryshevsky et al. Experimental observation of radiation frequency tuning in OLSE-10 prototype of volume free electron laser. *Nucl. Instr. Meth.*, **B252**:86–91, 2006.

[16] H.P. Freund. Time-dependent simulation of free-electron laser amplifiers and oscillators. *Phys. Rev. ST Accel. Beams*, **8**:110701, 2005.

[17] H.P. Freund, M. Shinn and S.V. Benson. Simulation of high-average power free-electron laser oscillator. *Phys. Rev. ST Accel. Beams*, **10**:030702, 2007.

[18] N.S. Ginzburg, R.M. Rosental and A.S. Sergeev. On the synthesis of radiation spectrum in a sectioned relativistic backward wave tube. *Tech. Phys. Lett.*, **29**:164–167, 2003.

[19] S.H. Gold and G.S. Nusinovich. Review of high-power microwave source research. *Rev. Sci. Instrum.*, **68**:3945–3974, 1997.

[20] S.L. Hahn and J.K. Lee. Nonlinear short-pulse propagation in free-electron laser. *Phys. Rev.*, **E48**:2162–2171, 1993.

[21] Z. Huang and K. Kim. Review of x-ray free-electron laser theory. *Phys. Rev. ST Accel. Beams*, **10**:034801, 2007.

[22] M.S. Hur, H.J. Lee and J.K. Lee. Parametrization of nonlinear and chaotic oscillations in driven beam-plasma diodes. *Phys. Rev.*, **E58**:936–941, 1998.

[23] S.P. Kuznetsov. Nonlinear dynamics of backward-wave tube: self-modulation, multi-stability, control. *Izvestija VUZov – Applied Nonlinear Dynamics*, **14**:3–35, 2006. (in Russian)

[24] S.P. Kuznetsov and D.I. Trubetskov. Chaos and hyper chaos in backward wave tube. *Izvestija VUZov – Radiophysics*, **47**:383–399, 2004. (in Russian)

[25] E.N. Lorenz. Deterministic nonperiodic flow. *J. Atmos. Sci.*, **20**:130–141, 1963.

[26] J.M.J. Madey. Stimulated emission of bremsstrahlung in a periodic magnetic field. *J. Appl. Phys.*, **42**:1906–1913, 1971.

[27] S. Reiche. GENESIS 1.3: a fully 3d time-dependent FEL simulation code. *Nucl. Instrum. Meth.*, **A429**:243–248, 1999.

[28] C.W. Roberson and P. Sprangle. A review of free-electron laser. *Phys. Fluids B*, **1**:3–41, 1989.

[29] H.-G. Schuster. *Deterministic Chaos. An Introduction.* Physik Verlag, Weinheim, 1984.

[30] S. Sytova. Volume Free Electron Laser (VFEL) as a dynamical system. *Nonlin. Phen. Compl. Syst.*, **10**(3):297–302, 2007.

Math. Model. Anal., 13(2):263–274, 2008.