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FINITE-DIFFERENCE SCHEME FOR ONE PROBLEM OF NONLINEAR OPTICS

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Abstract. We consider a mathematical model, which describes Q-switching process. The finite difference scheme is developed for approximation of the given system of nonlinear PDEs. It is constructed by using the staggered grid, such a strategy enables an automatic linearization of the algorithm. The transport equations are approximated along characteristics $z \pm t$, thus no discretization error is introduced at this stage. But such algorithm puts a strong relation between time and space steps of the discrete grid. The convergence analysis of this scheme is done using the method developed in [2]. First some estimates of the boundedness of the exact solution are proved. Then the boundedness of the discrete solution is investigated. On the basis of these estimates the main stability inequality is proved. The second order convergence rate with respect the space and time coordinates follows from this estimate.

Key words: finite difference method, nonlinear PDE, nonlinear optics, mathematical modelling.

1 Mathematical Model

The Q-switching is a technique which allows the production of light pulses with extremely high peak power. We consider a mathematical model, which describes the dynamics of two photon fluxes $I^{\pm} = I^{\pm}(z,t)$ propagating in the opposite directions. These fluxes interact inside the laser's resonator and they are coupled through boundary conditions and active medium. The gain evolution in a laser medium is described by function $N = N(z,t)$.

In this paper we consider a simplified mathematical model of Q-switched laser, which can be described by the following nonlinear system of differential equations [5, 6]:

$$\begin{cases} \dfrac{1}{V}\dfrac{\partial I^+}{\partial t} + \dfrac{\partial I^+}{\partial z} = \sigma\big(N - N_t\big)GI^+ - \alpha^+ I^+ + \dfrac{\beta^+ NG}{\tau_s}, & 0 < z \le Z_R, \\ \dfrac{1}{V}\dfrac{\partial I^-}{\partial t} - \dfrac{\partial I^-}{\partial z} = \sigma\big(N - N_t\big)GI^- - \alpha^- I^- + \dfrac{\beta^- NG}{\tau_s}, & 0 \le z < Z_R, \\ \dfrac{dN}{dt} = S(z,t) - \dfrac{(N - N_t)\big(I^+ + I^-\big)}{E_s} - \dfrac{N}{\tau_s}, & 0 \le z \le Z_R, \end{cases} \tag{1.1}$$

where V is the velocity of light propagation in an active medium, E_s is the saturation energy density describing the amount of energy that can be stored in a laser, N_t is the coefficient of linear losses, $S(z,t)$ is a source term. The parameters of the mathematical model satisfy the following requirements of the positivity or non-negativity

$$E_s, \tau_s > 0, \quad \sigma, \alpha^{\pm}, \beta^{\pm}, G, N_t \ge 0.$$

The initial conditions describe the initial distribution of fluxes and the gain function at the moment of the beginning of Q-switching process:

$$I^{\pm}(z,0) = I_0^{\pm}, \quad N(z,0) = N_0(z). \tag{1.2}$$

The boundary and conjugation conditions describe the influence of the reflectors at $z = 0$ and $z = Z_R$ and the transmission of the Q-switch at $z = Z_s < Z_R$.

$$I^+(0,t) = R_1 I^-(0,t) + f(t), \quad I^-(Z_R,t) = R_2 I^+(Z_R,t), \tag{1.3}$$

$$I^+(Z_s + 0,t) = T_s^+(t) I^+(Z_s,t), \quad I^-(Z_s - 0,t) = T_s^-(t) I^-(Z_s,t),$$

where R_1, R_2 are the reflectivities of the mirrors, and $T_s^{\pm}(t)$ are the transmissions of the Q-switch

$$0 \le R_1, R_2 \le 1, \quad 0 \le T_s^{\pm}(t) \le 1, \quad t > 0.$$

Function $f(t)$ defines a seed flux, which is small in comparison with the maximum value of the output laser flux. An example of $f(t)$ is described as:

$$f(t) = \begin{cases} A\Big(\dfrac{t}{\tau_p}\Big)^{\beta} e^{\beta(1 - t/\tau_p)}, & \beta > 0, \ \ 0 \le t \le \tau_p, \\ A, \quad t > \tau_p. \end{cases} \tag{1.4}$$

The rest of the paper is organized as follows. In Section 2 the finite difference scheme is developed for approximation of the given system of nonlinear PDEs. The convergence analysis of this scheme is done in Section 3. First some estimates of the boundedness of the exact solution are proved. Then the boundedness of the discrete solution is investigated. On the basis of these estimates the main stability inequality is proved. Some final conclusions are done in Section 4.

2 Finite-Difference Scheme

Let us now define a pair of uniform staggered grids $(\omega_h, \widetilde{\omega}_h)$, here $\omega_h = \omega_{h_z} \times \omega_{h_t}$ and $\widetilde{\omega}_h = \widetilde{\omega}_{h_z} \times \widetilde{\omega}_{h_t}$:

$$\omega_{h_z} = \Big\{ z_j : \ \ z_j = jh_z, \ \ 0 \le j \le J, \ \ z_J = Z_R \Big\},$$

$$\widetilde{\omega}_{h_z} = \Big\{ z_{j+1/2} : \ \ z_{j+1/2} = \Big(j + \frac{1}{2}\Big) h_z, \quad 0 \le j < J, \Big\},$$

$$\omega_{h_t} = \Big\{ t^n : \ \ t^n = nh_t, \ \ n = 0, 1, \dots, M, \ Mh_t = T \Big\},$$

$$\widetilde{\omega}_{h_t} = \Big\{ t^{n+1/2} : \ \ t^{n+1/2} = \Big(n + \frac{1}{2}\Big) h_t, \ \ n = 0, 1, \dots, M-1 \Big\}.$$

On these grids discrete functions are defined as

$$U_j^{\pm,n} = U^{\pm}(z_j, t^n), \quad K_{j+1/2}^{n+1/2} = K(z_{j+1/2}, t^{n+1/2}),$$

here $U_j^{\pm,n}$ approximates $I^{\pm}(z,t)$ and $K_{j+1/2}^{n+1/2}$ approximates the gain function $N(z,t)$. We discretize system (1.1) by using the finite-difference method [4]:

$$\begin{cases} \dfrac{U_j^{+,n} - U_{j-1}^{+,n-1}}{h_z} = \big[\sigma G\big(K_{j-1/2}^{n-1/2} - N_t\big) - \alpha^+\big] \dfrac{U_j^{+,n} + U_{j-1}^{+,n-1}}{2} \\ \qquad\qquad + \dfrac{\beta^+ G}{\tau_s} K_{j-1/2}^{n-1/2}, \\ \dfrac{U_{j-1}^{-,n} - U_j^{-,n-1}}{h_z} = \big[\sigma G\big(K_{j-1/2}^{n-1/2} - N_t\big) - \alpha^-\big] \dfrac{U_{j-1}^{-,n} + U_j^{-,n-1}}{2} \\ \qquad\qquad + \dfrac{\beta^- G}{\tau_s} K_{j-1/2}^{n-1/2}, \\ \dfrac{K_{j-1/2}^{n+1/2} - K_{j-1/2}^{n-1/2}}{h_t} = \dfrac{S_{j-1/2}^{n+1/2} + S_{j-1/2}^{n-1/2}}{2} - \left(\dfrac{K_{j-1/2}^{n+1/2} + K_{j-1/2}^{n-1/2}}{2} - N_t \right) \\ \qquad \times \Big(\dfrac{1}{2E_s} \sum\limits_{k=j-1}^{j} (U_k^{+,n} + U_k^{-,n}) \Big) - \dfrac{K_{j-1/2}^{n+1/2} + K_{j-1/2}^{n-1/2}}{2\tau_s}. \end{cases} \tag{2.1}$$

Here the convection terms are approximated along characteristics, therefore we take $h_z = Vh_t$. A time integration is implemented by using the Crank-Nicolson method. The staggered grids for numerical approximation of parabolic diffusion-reaction equations were used in many papers, e.g. for solution of the Hodgkin-Huxley type equations [1, 3].

The discrete boundary and initial conditions are defined as:

$$U_0^{+,n} = R_1 U_0^{-,n} + f(t^n), \quad U_J^{-,n} = R_2 U_J^{+,n}, \quad t^n \in \omega_{h_t}, \tag{2.2}$$

$$U_j^{\pm,0} = I_0^{\pm}(z_j), \ \ z_j \in \omega_{h_z}, \quad K_{j-1/2}^0 = N_0(z_{j-1/2}), \quad z_{j-1/2} \in \widetilde{\omega}_{h_z}.$$

We compute the values $K^{1/2}_{j-1/2}$ at the first step of the staggered time grid by using the linearized Euler integration method for $z_{j-1/2} \in \widetilde{\omega}_{h_z}$:

$$\frac{K^{1/2}_{j-1/2} - K^0_{j-1/2}}{0.5h_t} = S^{1/2}_{j-1/2} - \frac{K^{1/2}_{j-1/2} - N_t}{2E_s} \sum_{k=j-1}^{j} (U_k^{+,0} + U_k^{-,0}) - \frac{K^{1/2}_{j-1/2}}{\tau_s}. \tag{2.3}$$

At the transmission point of the Q-switch $z_S = Z_s$ we change the approximation of the transport equations taking into account the conjugation conditions:

$$\frac{U^{+,n}_{S+1} - T^+_s(t)U^{+,n-1}_S}{h_z} = \left[\sigma G\left(K^{n-1/2}_{j-1/2} - N_t\right) - \alpha^+\right] \frac{U^{+,n}_{S+1} + T^+_s(t)U^{+,n-1}_S}{2} + \frac{\beta^+ G}{\tau_s} K^{n-1/2}_{j-1/2}, \tag{2.4}$$

$$\frac{U^{-,n}_{S-1} - T^-_s(t)U^{-,n-1}_S}{h_z} = \left[\sigma G\left(K^{n-1/2}_{j-1/2} - N_t\right) - \alpha^-\right] \frac{U^{-,n}_{S-1} + T^-_s(t)U^{-,n-1}_S}{2} + \frac{\beta^- G}{\tau_s} K^{n-1/2}_{j-1/2}.$$

The implementation of finite difference scheme (2.1)–(2.4) is explicit. First the transport equations are solved for $j = 1, 2, \ldots, J$:

$$U^{+,n}_j = \begin{cases} \dfrac{\left(1 + 0.5h_z D^{+,n-1/2}_{j-1/2}\right)U^{+,n-1}_{j-1} + h_z B^+ K^{n-1/2}_{j-1/2}}{1 - 0.5h_z D^{+,n-1/2}_{j-1/2}}, & j \neq S+1, \\[2ex] \dfrac{\left(1 + 0.5h_z D^{+,n-1/2}_{j-1/2}\right)T^+_s U^{+,n-1}_{j-1} + h_z B^+ K^{n-1/2}_{j-1/2}}{1 - 0.5h_z D^{+,n-1/2}_{j-1/2}}, & j = S+1, \end{cases} \tag{2.5}$$

$$U^{-,n}_{j-1} = \begin{cases} \dfrac{\left(1 + 0.5h_z D^{-,n-1/2}_{j-1/2}\right)U^{-,n-1}_{j} + h_z B^- K^{n-1/2}_{j-1/2}}{1 - 0.5h_z D^{-,n-1/2}_{j-1/2}}, & j \neq S, \\[2ex] \dfrac{\left(1 + 0.5h_z D^{-,n-1/2}_{j-1/2}\right)T^-_s U^{-,n-1}_{j} + h_z B K^{n-1/2}_{j-1/2}}{1 - 0.5h_z D^{n-1/2}_{-,j-1/2}}, & j = S, \end{cases}$$

where notations $D^{\pm,n-1/2}_{j-1/2} = \sigma G\left(K^{n-1/2}_{j-1/2} - N_t\right) - \alpha^{\pm}$, $B^{\pm} = \dfrac{\beta^{\pm} G}{\tau_s}$ are used.

In the second step, by using the third equation in (2.1), we find new values of the gain function

$$K_{j-1/2}^{n+1/2} = \left(\frac{1-0.5h_t L_{j-1/2}^n}{1+0.5h_t L_{j-1/2}^n}\right) K_{j-1/2}^{n-1/2} + \frac{0.5h_t\left(S_{j-1/2}^{n+1/2} + S_{j-1/2}^{n-1/2}\right)}{1+0.5h_t L_{j-1/2}^n} + \frac{h_t N_t\left(L_{j-1/2}^n - \frac{1}{\tau_s}\right)}{1+0.5h_t L_{j-1/2}^n}, \quad j = 1,2,\dots,J, \tag{2.6}$$

where the following notation

$$L_{j-1/2}^n = \frac{1}{2E_s}\sum_{k=j-1}^{j}(U_k^{+,n} + U_k^{-,n}) + \frac{1}{\tau_s}, \quad n \ge 0.$$

is used.

3 Convergence Analysis

3.1 The boundedness of the solution of (1.1)

In this section we investigate the boundedness of functions $N(z,t)$ and $I^{\pm}(z,t)$. Let us assume, that the initial data satisfies the following inequalities

$$0 \le N_0(z) \le C_{N0}, \quad 0 \le I_0^{\pm}(z) \le C_{I0}, \quad 0 \le z \le Z_R. \tag{3.1}$$

The source term in the gain equation is also assumed to be non-negative and bounded from above:

$$0 \le S(z,t) \le C_S, \quad 0 \le z \le Z_R, \;\; t > 0. \tag{3.2}$$

It follows from (1.4) that

$$0 \le f(t) \le A, \quad t > 0. \tag{3.3}$$

Lemma 1. *If conditions* (3.1)–(3.3) *are satisfied then for a solution of system* (1.1)–(1.3) *the following estimates are valid:*

$$I^{\pm}(z,t) \ge 0, \quad 0 \le N(z,t) \le C_N, \quad 0 \le z \le Z_R, \;\; t > 0.$$

Proof. By using a special structure of equations (1.1), the positivity of $S(z,t)$ and applying the classical fixed-point iteration technique it is easy to prove that all solutions are non-negative

$$I^{\pm}(z,t) \ge 0, \quad N(z,t) \ge 0, \quad 0 \le z \le Z_R, \;\; t > 0.$$

Thus it remains to prove the boundedness of $N(z,t)$ from above. Let us rewrite the initial value problem for the gain function as

$$\begin{cases} \dfrac{d(N(z,t)-N_t)}{dt} + \Big(\dfrac{I^+ + I^-}{E_s} + \dfrac{1}{\tau_s}\Big)(N(z,t)-N_t) = S(z,t) - \dfrac{N_t}{\tau_s}, \\ N(z,0) - N_t = N_0(z) - N_t. \end{cases}$$

We put the solution in the form $N(z,t) = N_t + v(z,t) + w(z,t)$, where $v(z,t)$ solves the homogeneous initial value problem

$$\begin{cases} \dfrac{dv}{dt} + \Big(\dfrac{I^+ + I^-}{E_s} + \dfrac{1}{\tau_s}\Big) v = 0, \\ v(z,0) = N_0(z) - N_t, \end{cases}$$

and $w(z,t)$ is the solution of the non-homogeneous initial value problem subject to zero initial conditions

$$\begin{cases} \dfrac{dw}{dt} + \Big(\dfrac{I^+ + I^-}{E_s} + \dfrac{1}{\tau_s}\Big) w = S(z,t) - \dfrac{N_t}{\tau_s}, \\ w(z,0) = 0. \end{cases}$$

Taking into account that $I^{\pm}(z,t) \geq 0$ we get the estimate

$$v(z,t) \leq \max(0, C_{N0} - N_t). \tag{3.4}$$

Function $w(z,t)$ can be bounded from above by $\widetilde{w}(t)$ satisfying the initial value problem

$$\begin{cases} \dfrac{d\widetilde{w}}{dt} + \dfrac{1}{\tau_s}\widetilde{w} = C_S - \dfrac{N_t}{\tau_s}, \\ \widetilde{w}(z,0) = 0. \end{cases}$$

The solution of this problem is given by

$$\widetilde{w}(t) = (\tau_s C_S - N_t)(1 - e^{-t/\tau_s}),$$

thus the following estimates are valid:

$$w(z,t) \leq \widetilde{w}(t) \leq \max(0, \tau_s C_S - N_t). \tag{3.5}$$

From estimates (3.4) and (3.5) we get the boundedness estimate of function $N(z,t)$:

$$N(z,t) \leq \max(N_t, C_{N0}) + \max(0, \tau_s C_S - N_t).$$

□

The dynamical stability of the whole system and as a consequence the boundedness from above of functions $I^{\pm}(z,t)$ depends on the parameters of the problem (mainly, on the reflectivity coefficients at the boundaries of the space domain) and amount of energy, pumped into the reflector. Let us assume that functions $I^{\pm}(z,t)$ are uniformly bounded from above by some constant

$$I^{\pm}(z,t) \leq C_I, \quad 0 \leq z \leq Z_R, \ \ t > 0.$$

3.2 The boundedness of the discrete solution

In this section we investigate the boundedness of the discrete solution of finite difference scheme (2.1) – (2.4).

We start our analysis assuming that

$$U_j^{\pm} \le C_{Ih}, \quad j = 0, 1, \ldots, J, \tag{3.6}$$

where constant C_{Ih} can be selected as $C_{Ih} = C_I + 1$. This assumption will be justified later, when the main stability estimate will be obtained. Such a method of the analysis of nonlinear difference schemes was developed in [1, 2].

Lemma 2. *If conditions* (3.1)–(3.3), (3.6) *are satisfied then for a sufficiently small time step* $h_t \le h_t^0$ *the apriori estimates*

$$0 \le K_{j+1/2}^{n+1/2} \le C_{Nh}, \quad U_j^{\pm,n} \ge 0, \quad n = 0, 1, \ldots, M, \;\; j = 0, 1, \ldots, J \tag{3.7}$$

are valid for the solution of discrete problem

Proof. First we show that estimates (3.7) are valid at the initial time moment $n = 0$. The non-negativity of $U_j^{\pm,0}$ follows from the initial condition and the assumptions of this lemma. Repeating the proof of Lemma 1 we get from (2.3) that the estimates

$$0 \le K_{j-1/2}^{1/2} \le C_N, \quad j = 1, 2, \ldots, J$$

are valid unconditionally.

Let us assume that estimates (3.7) are valid at the time moment t^{n-1}, where $n \le M$. Since $T_s^{\pm} \ge 0$ and taking a sufficiently small time step

$$h_t < h_{t,1}^0 = \frac{2}{\sigma G\left(C_{Nh} - N_t\right) - \max(\alpha^+, \alpha^-)},$$

we get from (2.5) that

$$U_j^{\pm,n} \ge 0, \;\; z_j \in \omega_z.$$

This restriction on the time step h_t arises due to the conditional monotonicity of the symmetrical Crank-Nicolson approximation.

It follows from (2.6) that for a sufficiently small time step $h_t \le h_{t,2}^0$, where $h_{t,2}^0 = 2E_s\tau_s/(2C_{Ih}\tau_s + E_s)$ the estimates

$$K_{j-1/2}^{n+1/2} \ge 0, \quad j = 1, 2, \ldots, J$$

are valid. Thus it remains to prove that $K_{j-1/2}^{n+1/2}$ is bounded from above by a constant not depending on n. Let us rewrite the discrete problem for the gain function as

$$
\begin{cases}
\dfrac{(K_{j-1/2}^{n+1/2}-N_t)-(K_{j-1/2}^{n-1/2}-N_t)}{h_t}+\dfrac{1}{2}\big[(K_{j-1/2}^{n+1/2}-N_t)+(K_{j-1/2}^{n-1/2}-N_t)\big] \\
\qquad \times\dfrac{1}{2E_s}\sum\limits_{k=j-1}^{j}(U_k^{+,n}+U_k^{-,n})+\dfrac{(K_{j-1/2}^{n+1/2}-N_t)+(K_{j-1/2}^{n-1/2}-N_t)}{2\tau_s} \\
\qquad =\dfrac{S_{j-1/2}^{n+1/2}+S_{j-1/2}^{n-1/2}}{2}-\dfrac{N_t}{\tau_s}.
\end{cases}
$$

We put the solution in the form $K_{j-1/2}^{n+1/2}=N_t+V_{j-1/2}^{n+1/2}+W_{j-1/2}^{n+1/2}$, where $V_{j-1/2}^{n+1/2}$ solves the homogeneous discrete problem

$$
\begin{cases}
\dfrac{V_{j-1/2}^{n+1/2}-V_{j-1/2}^{n-1/2}}{h_t}+\dfrac{V_{j-1/2}^{n+1/2}+V_{j-1/2}^{n-1/2}}{2}\Big(\dfrac{1}{2E_s}\sum\limits_{k=j-1}^{j}(U_k^{+,n}+U_k^{-,n})+\dfrac{1}{\tau_s}\Big) \\
\qquad =0,\quad n>0,\ \ j=1,2,\dots,J, \\
V_{j-1/2}^{1/2}=K_{j-1/2}^{1/2}-N_t,\quad j=1,2,\dots,J,
\end{cases}
$$

and $W_{j-1/2}^{n+1/2}$ solves the non-homogeneous discrete problem with zero initial conditions:

$$
\begin{cases}
\dfrac{W_{j-1/2}^{n+1/2}-W_{j-1/2}^{n-1/2}}{h_t}+\dfrac{W_{j-1/2}^{n+1/2}+W_{j-1/2}^{n-1/2}}{2}\Big(\dfrac{1}{2E_s}\sum\limits_{k=j-1}^{j}(U_k^{+,n}+U_k^{-,n})+\dfrac{1}{\tau_s}\Big) \\
\qquad =\dfrac{S_{j-1/2}^{n+1/2}+S_{j-1/2}^{n-1/2}}{2}-\dfrac{N_t}{\tau_s},\quad n>0,\ \ j=1,2,\dots,J, \\
W_{j-1/2}^{1/2}=0,\quad j=1,2,\dots,J.
\end{cases}
$$

Taking a sufficiently small time step $h_t\le h_{t,2}^0$ and using the estimate from above for $K_{j-1/2}^{1/2}$ we prove that $V_{j-1/2}^{n+1/2}\le C_N$.

Since $(1-x)/(1+x)$ is a monotonically decreasing function, then the following estimate is valid for $h_t\le h_{t,2}^0$:

$$
W_{j-1/2}^{n+1/2}\le\frac{1-0.5h_t/\tau_s}{1+0.5h_t/\tau_s}W_{j-1/2}^{n-1/2}+\frac{C_Bh_t}{1+0.5h_t/\tau_s},\quad C_B=\max(0,C_S-N_t/\tau_s).
$$

By iterating this inequality we get

$$
\begin{aligned}
W_{j-1/2}^{n+1/2}&\le\frac{C_Bh_t}{1+0.5h_t/\tau_s}\left(1+\frac{1-0.5h_t/\tau_s}{1+0.5h_t/\tau_s}+\ldots+\Big(\frac{1-0.5h_t/\tau_s}{1+0.5h_t/\tau_s}\Big)^n\right) \\
&\le C_B\tau_s=\max(0,C_S\tau_s-N_t).
\end{aligned}
$$

Adding all estimates we get the estimate from above for $K_{j-1/2}^{n+1/2}$:

$$K_{j-1/2}^{n+1/2} = C_N + \max(N_t, C_S\tau_s), \quad j = 1, 2, \dots, J.$$

Finally, we take $h_t^0 = \min(h_{t,1}^0, h_{t,2}^0)$. Thus it is proved that both apriori estimates (3.7) are satisfied at the next time moment t^n. The proof is concluded by induction on n. □

3.3 Stability analysis

In this section we investigate the stability and convergence of difference scheme (2.1)–(2.4). Let us denote the error functions of the discrete solution:

$$Z_{j+1/2}^{n+1/2} = K_{j+1/2}^{n+1/2} - N(z_{j+1/2}, t^{n+1/2}), \quad P_j^{\pm,n} = U_j^{\pm,n} - I^{\pm}(z_j, t^n).$$

By putting these functions into the finite-difference scheme we get a discrete problem for the error functions.

3.3.1 Analysis of the gain equation

First we investigate the equation for the gain function:

$$\frac{Z_{j-1/2}^{n+1/2} - Z_{j-1/2}^{n-1/2}}{h_t} + \frac{Z_{j-1/2}^{n+1/2} + Z_{j-1/2}^{n-1/2}}{2} \frac{\sum\limits_{l=j-1}^{j} (U_l^{+,n} + U_l^{-,n})}{2E_s}$$
$$+ \frac{Z_{j-1/2}^{n+1/2} + Z_{j-1/2}^{n-1/2}}{2\tau_s} = -\frac{N_{j-1/2}^{n+1/2} + N_{j-1/2}^{n-1/2}}{4E_s} \sum_{l=j-1}^{j} (P_l^{+,n} + P_l^{-,n}) - R_{j-1/2}^n,$$

where $R_{j-1/2}^n$ denotes the residual of the discrete equation:

$$R_{j-1/2}^n = \frac{N_{j-1/2}^{n+1/2} - N_{j-1/2}^{n-1/2}}{h_t} + \Big(\frac{N_{j-1/2}^{n+1/2} + N_{j-1/2}^{n-1/2}}{2} - N_t\Big) \frac{\sum\limits_{l=j-1}^{j} (I_l^{+,n} + I_l^{-,n})}{2E_s}$$
$$+ \frac{N_{j-1/2}^{n+1/2} + N_{j-1/2}^{n-1/2}}{2\tau_s} - \frac{S_{j-1/2}^{n+1/2} + S_{j-1/2}^{n-1/2}}{2}.$$

The approximation error is estimated as

$$\|R^n\|_\infty \le C(h_z^2 + h_t^2).$$

By using the positivity of functions $U^{\pm}$ and boundedness of $N^{n+1/2}$ we get the stability inequality

$$\big|Z_{j-1/2}^{n+1/2}\big| \le \big|Z_{j-1/2}^{n-1/2}\big| + \frac{C_N h_t}{E_s}(\|P^{+,n}\|_\infty + \|P^{-,n}\|_\infty) + h_t\|R^n\|_\infty.$$

It remains to estimate the error at the first time step. Similarly to analysis given above and taking into account that initial conditions are exact

$$P_j^{\pm,0} = 0, \;\; z_j \in \omega_{h_z}, \quad Z_{j-1/2}^0 = 0, \;\; z_{j-1/2} \in \tilde{\omega}_{h_z},$$

we get the equation for the error function:

$$\frac{2}{h_t} Z_{j-1/2}^{1/2} + \Big(\frac{1}{2E_s} \sum_{l=j-1}^{j} (I_l^{+,0} + I_l^{-,0}) + \frac{1}{\tau_s} \Big) Z_{j-1/2}^{1/2} = -R_{j-1/2}^0,$$

where $R_{j-1/2}^0$ denotes the residual of the discrete equation:

$$R_{j-1/2}^0 = \frac{N_{j-1/2}^{1/2} - N_{j-1/2}^0}{0.5 h_t} + \frac{N_{j-1/2}^{1/2}}{\tau_s} - S_{j-1/2}^{1/2}$$
$$+ \Big(\frac{N_{j-1/2}^{1/2} + N_{j-1/2}^0}{2} - N_t \Big) \frac{1}{2E_s} \sum_{l=j-1}^{j} (I_l^{+,0} + I_l^{-,0}).$$

The approximation error is estimated as

$$\|R^0\|_\infty \le C(h_z^2 + h_t).$$

By using the positivity of functions $I^{\pm}$ we get the stability inequality

$$\big| Z_{j-1/2}^{1/2} \big| \le \frac{h_t}{2} \|R^0\|_\infty \le C(h_t^2 + h_t h_z^2).$$

3.3.2 Analysis of the transport equations

Next we investigate equations for the error functions $P^{\pm}$. Since this analysis is similar for both functions, we will present a detailed analysis only for the forward moving wave function. The error function satisfies the following equations:

$$\frac{P_j^{+,n} - P_{j-1}^{+,n-1}}{h_z} = \big[\sigma G \big(K_{j-1/2}^{n-1/2} - N_t \big) - \alpha^+ \big] \frac{P_j^{+,n} + P_{j-1}^{+,n-1}}{2}$$
$$+ \sigma G \, Z_{j-1/2}^{n-1/2} \frac{I_j^{+,n} + I_{j-1}^{+,n-1}}{2} + \frac{\beta^+ G}{\tau_s} Z_{j-1/2}^{n-1/2} - R_{u,j-1/2}^{+,n-1/2},$$
$$j = 1, 2, \ldots, J, \; j \ne S+1,$$

$$\frac{P_{S+1}^{+,n} - T_S^+(t) P_S^{+,n-1}}{h_z} = \big[\sigma G \big(K_{S+1/2}^{n-1/2} - N_t \big) - \alpha^+ \big] \frac{P_{S+1}^{+,n} + T_S^+(t) P_S^{+,n-1}}{2}$$
$$+ \sigma G \, Z_{S+1/2}^{n-1/2} \frac{I_{S+1}^{+,n} + T_S^+(t) I_S^{+,n-1}}{2} + \frac{\beta^+ G}{\tau_s} Z_{S+1/2}^{n-1/2} - R_{u,S+1/2}^{+,n-1/2}, \; j = S+1,$$

where $R^{+,n-1/2}_{u,j-1/2}$ denotes the residual of the discrete equation

$$R^{+,n-1/2}_{u,j-1/2} = \frac{I^{+,n}_j - I^{+,n-1}_{j-1}}{h_z} - \big[\sigma G\big(N^{n-1/2}_{j-1/2} - N_t\big) - \alpha^+\big] \frac{I^{+,n}_j + I^{+,n-1}_{j-1}}{2}$$
$$- \frac{\beta^+ G}{\tau_s} N^{n-1/2}_{j-1/2}, \quad j = 1, 2, \dots, J,\ j \neq S+1,$$
$$R^{+,n-1/2}_{u,S+1/2} = \frac{I^{+,n}_{S+1} - T^+_S(t) I^{+,n-1}_S}{h_z} - \big[\sigma G\big(N^{n-1/2}_{S+1/2} - N_t\big) - \alpha^+\big]$$
$$\times \frac{I^{+,n}_{S+1} + T^+_S(t) I^{+,n-1}_S}{2} - \frac{\beta^+ G}{\tau_s} N^{n-1/2}_{S+1/2}, \quad j = S+1.$$

The approximation error is estimated as

$$\|R^{+,n-1/2}_u\|_\infty \le C(h_z^2 + h_t^2).$$

For $j = 1, 2, \dots, J$ we write the the discrete error equation as

$$P^{+,n}_j = \left(\frac{1 + 0.5 h_z D^{n-1/2}_{j-1/2}}{1 - 0.5 h_z D^{n-1/2}_{j-1/2}}\right) P^{+,n-1}_{j-1} + \frac{h_z}{1 - 0.5 h_z D^{n-1/2}_{j-1/2}}$$
$$\times \left(\sigma G \left(\frac{I^{+,n}_j + I^{+,n-1}_{j-1}}{2} + \frac{\beta^+}{\sigma \tau_s}\right) Z^{n-1/2}_{j-1/2} - R^{+,n-1/2}_{u,j-1/2}\right).$$

By taking a sufficiently small $h_z \le 1/(3C_{Nh})$ and denoting $C_I = \|I^+\|_{C(0,Z)}$ we get the following estimate

$$\big|P^{+,n}_j\big| \le (1 + 3C_{Nh} h_z)\big|P^{+,n-1}_{j-1}\big| + \frac{6h_z}{5}\Big(C_I\big|Z^{n-1/2}_{j-1/2}\big| + \big|R^{+,n-1/2}_{u,j-1/2}\big|\Big), \ j = 1, \dots, J.$$

Similar estimates are valid for function $P^{-,n}$:

$$\big|P^{-,n}_{j-1}\big| \le (1 + 3C_{Nh} h_z)\big|P^{-,n-1}_j\big| + \frac{6h_z}{5}\Big(C_I\big|Z^{n-1/2}_{j-1/2}\big| + \big|R^{-,n-1/2}_{u,j-1/2}\big|\Big), \ j = 1, \dots, J.$$

We get from the boundary conditions that

$$\big|P^{+,n}_0\big| \le |R_1|\,\big|P^{-,n}_0\big|, \quad \big|P^{-,n}_J\big| \le |R_2|\,\big|P^{+,n}_J\big|.$$

Let us denote

$$\big\|P^n\big\|_\infty = \max\big(\big\|P^{+,n}\big\|_\infty, \big\|P^{-,n}\big\|_\infty\big).$$

Combining the inequalities given above and taking into account that $|R_j| \le 1$ and $h_z = V h_t$, we prove the second stability estimate

$$\big\|P^n\big\|_\infty \le (1 + 3C_{Nh} V h_t)\big\|P^{n-1}\big\|_\infty + \frac{6Vh_t}{5}\Big(C_I\big\|Z^{n-1/2}\big\|_\infty + \big\|R^{n-1/2}_u\big\|_\infty\Big).$$

4 Conclusions

A simplified mathematical model of Q-switching is approximated by the symmetric finite difference scheme. This approximation is constructed on the staggered grid which enables automatic linearization of the discrete problem. The convergence analysis is based on a priori boundedness estimates of the discrete solution. Such estimates give an important information on the numerical solution, establishing a connection with similar properties of the exact solution.

The main goal of a future work is to generalize the obtained results for 2D nonstationary mathematical models defining the problem in (z, x) space domain.

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