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ON NONLINEAR FUČIK TYPE SPECTRA

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Abstract. Eigenvalue problems of the form $x'' = -\lambda f(x^+) + \mu g(x^-)$, $x(0) = 0$, $x(1) = 0$ are considered, where x^+ and x^- are respectively the positive and the negative parts of x. We are looking for (λ, μ) such that the problem has a nontrivial solution. This problem generalizes the famous Fučik problem for piece-wise linear equations. In order to show that nonlinear Fučik spectra may differ essentially from the classical ones, we consider piece-wise linear functions $f(x)$ and $g(x)$. We show that the first branches of the Fučik spectrum may contain bounded components.

Key words: Fučik spectra, nonlinear boundary value problem.

1 Introduction

In [3] we have considered the two-parameter eigenvalue problem

$$x'' = -\lambda f(x^+) + \mu g(x^-), \tag{1.1}$$

$$x(0) = 0, \; x(1) = 0, \tag{1.2}$$

under the assumptions that f and g are continuous positive valued functions such that $f(0) = g(0) = 0$.

The same problem written in a usual form is given by

$$x'' = \begin{cases} -\lambda f(x), & \text{if} \quad x \geq 0 \\ \mu g(-x), & \text{if} \quad x < 0. \end{cases}$$

Obviously λ and μ must be non-negative in order the problem to have nontrivial solutions. If f and g are linear then (1.1) becomes the famous Fučik equation

$$x'' = -\lambda x^+ + \mu x^-, \tag{1.3}$$

where $x^+ = \max\{x, 0\}, \; x^- = \max\{-x, 0\}$.

This equation may be written also as

$$x'' = \begin{cases} -\lambda x, & \text{if} \quad x \geq 0 \\ -\mu x, & \text{if} \quad x < 0. \end{cases}$$

The Fuchik spectrum for the problem (1.3), (1.2) is defined as a set of all pairs (λ, μ) such that the problem has a nontrivial solution. This spectrum is well known ([2, § 35]) and consist of a set of a hyperbola looking curves located in the first quadrant and possessing vertical and horizontal asymptotes.

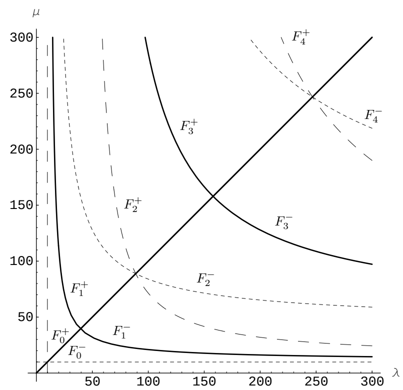


Figure 1. The classical Fučik spectrum.

We show that the Fučik type spectrum for the problem (1.1), (1.2) may differ essentially from the classical one. Namely, “branches” of the spectrum may be multicomponent, and the components may be infinite and even finite.

2 Description of the Spectrum

We consider the problem (1.1), (1.2) under the normalization condition

$$|x'(0)| = 1.$$

This condition must be imposed in order to avoid continuous spectra. For discussion see [3] and [1].

We assume that $f(x)$ satisfies the following condition:
(A1) A first zero $t_1(\alpha)$ of a solution to the Cauchy problem

$$u'' = -f(u), \quad u(0) = 0, \; u'(0) = \alpha$$

exists for any $\alpha > 0$.

A similar property can be assigned to $g(x)$. We assume that $g(x)$ satisfies the condition:
(A2) A first zero $\tau_1(\beta)$ of a solution to the Cauchy problem

$$v'' = g(-v), \quad v(0) = 0, \; v'(0) = -\beta$$

exists for any $\beta > 0$.

Let us recall (for the reader's convenience) the main result in [3]. Consider the problem

$$x'' = \begin{cases} -\lambda f(x), & \text{if} \quad x \geq 0 \\ \mu g(-x), & \text{if} \quad x < 0, \end{cases} \tag{2.1}$$

$$x(0) = x(1) = 0, \quad |x'(0)| = 1.$$

Theorem 1. *Let the conditions* **(A1)** *and* **(A2)** *hold with respect to the functions* $t_1(\gamma)$ *and* $\tau_1(\delta)$. *The Fuchik spectrum for the problem* (2.1) *is given by the relations* $(i = 1, 2, \ldots)$:

$$F_0^+ = \Big\{(\lambda, \mu) : \lambda \textit{ is a solution of } \frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1, \quad \mu \geq 0\Big\},$$

$$F_0^- = \Big\{(\lambda, \mu) : \lambda \geq 0,\ \mu \textit{ is a solution of } \frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\Big\},$$

$$F_{2i-1}^+ = \Big\{(\lambda; \mu) : \quad i\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + i\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\Big\},$$

$$F_{2i-1}^- = \Big\{(\lambda; \mu) : \quad i\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) + i\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1\Big\},$$

$$F_{2i}^+ = \Big\{(\lambda; \mu) : \quad (i+1)\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + i\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\Big\},$$

$$F_{2i}^- = \Big\{(\lambda; \mu) : \quad (i+1)\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) + i\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1\Big\}.$$

3 Equation $x'' = -\lambda f(x^+) + \mu f(x^-)$

Consider equation

$$x'' = -\lambda f(x^+) + \mu f(x^-), \tag{3.1}$$

together with the boundary conditions

$$x(0) = x(1) = 0, \quad |x'(0)| = 1, \tag{3.2}$$

Then

$$t_1(\gamma) = \tau_1(\gamma), \quad \gamma > 0. \tag{3.3}$$

Let

$$U(\lambda) := \frac{1}{\sqrt{\lambda}} t_1\Big(\frac{1}{\sqrt{\lambda}}\Big), \quad V(\mu) : \frac{1}{\sqrt{\mu}} \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big).$$

In view of (3.3) the spectrum of the problem (3.1), (3.2) is a union of the sets defined by the relations

$$F_1^{\pm} : U(\lambda) + U(\mu) = 1, \qquad F_2^+ :\ 2U(\lambda) + U(\mu) = 1,$$

$$F_2^- : U(\lambda) + 2U(\mu) = 1, \qquad F_3^{\pm} :\ 2U(\lambda) + 2U(\mu) = 1,$$

$$F_4^+ : 3U(\lambda) + 2U(\mu) = 1, \qquad F_4^- :\ 2U(\lambda) + 3U(\mu) = 1, \ldots$$

The coefficients at the first and second addends refer to the numbers of "positive" and "negative" humps of the respective eigenfunctions.

It is possible that the functions $U(\lambda)$ and $V(\mu)$ are non-monotone. Then spectra may differ essentially from those in the monotone case.

Remark 1. Suppose that for some positive integer p the functions $p\,U(\lambda)$ and $p\,V(\mu)$ are monotonically decreasing starting with some λ_* and μ_*, $p\,U(\lambda_*) = 1$, $p\,V(\mu_*) = 1$. Then branches $F^{\pm}_{2p-1}$ and higher behave like those in the monotone case. Generally $U(\lambda)$ and $V(\mu)$ are functions that tend to $+\infty$ as $\lambda, \mu \to 0+$ and $U, V \to 0+$ as $\lambda, \mu \to +\infty$. It easily can be shown that the respective spectra are structurally the same as the classical Fučik spectrum if $U(\lambda)$ and $V(\mu)$ are monotonically decreasing functions. Suppose that U is a function which is non-monotone and has exactly three successive intervals of monotonicity like shown in Fig. 2.

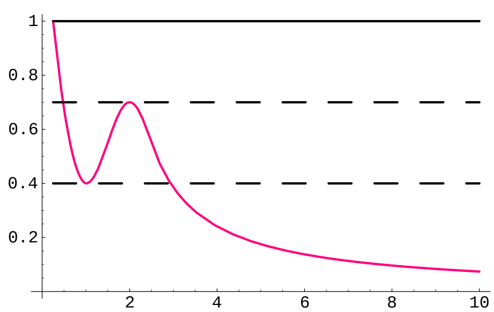


Figure 2. Function $U(\lambda)$, A and B are respectively the local minimum and maximum marked with dashed lines.

Let us analyze the subset $F^{\pm}_1$ of the spectrum. This subset is defined by the relation $U(\lambda) + U(\mu) = 1$. Denote the local minimum and local maximum of $U(\lambda)$ by A and B respectively.

Theorem 2. *Suppose that $U(\lambda)$ is a continuous positive valued function which satisfies the following conditions:*

- $U(\lambda) \to +\infty$ *as* $\lambda \to 0+$;
- $U(\lambda) \to 0+$ *as* $\lambda \to +\infty$;
- $U(\lambda)$ *has three intervals of strict monotonicity, as shown in Fig. 2;*
- *the local minimum A and the local maximum B of $U(\lambda)$ are such that* $A + B > 1$ *and* $A < 0.5 < B < 1$.

Then the subset $F^{\pm}_1$ of the Fučik spectrum consists of two disjoint sets (components), one of them being bounded.

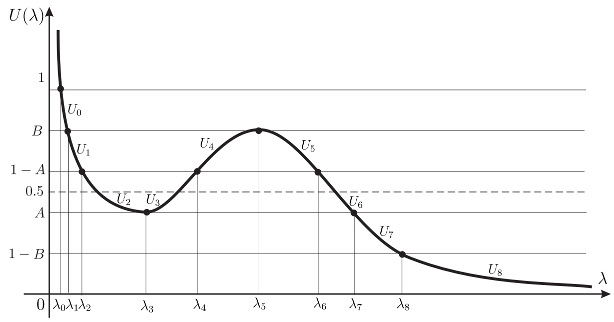


Figure 3. Subdivision of the λ-axis.

Proof. Consider the function U and introduce subdivision of λ-axis, as shown in Fig. 3. There are two local extrema at the points λ_3 and λ_5. The intervals of monotonicity and the respective ranges of values of the function are: the interval of decrease $I_1 = (0; \lambda_3]$, $U(I_1) = [A; +\infty)$; the interval of increase $I_2 = [\lambda_3; \lambda_5]$, $U(I_2) = [A; B]$; the interval of decrease $I_3 = [\lambda_5; +\infty)$, $U(I_3) = (0; B]$.

Notice also that $0 < 1 - B < A < 0.5 < 1 - A < B < 1$. By definition

$$U(\lambda_0) = 1, \;\; U(\lambda_1) = B, \;\; U(\lambda_2) = 1 - A,$$

$$U(\lambda_4) = 1 - A, \;\; U(\lambda_6) = 1 - A, \;\; U(\lambda_7) = A, \;\; U(\lambda_7) = 1 - B.$$

Consider the monotone restrictions $U_i(\lambda)$ $(\lambda = -1, 0, 1 \ldots, 8)$ of the function $U(\lambda)$:

Table 1.

Restriction	Interval	Range of values	Description
U_{-1}	$D_{-1} = (0; \lambda_0]$	$E_{-1} = [1; +\infty)$	decreasing
U_0	$D_0 = (\lambda_0; \lambda_1]$	$E_0 = [B; 1)$	decreasing
U_1	$D_1 = [\lambda_1; \lambda_2]$	$E_1 = [1 - A; B]$	decreasing
U_2	$D_2 = [\lambda_2; \lambda_3]$	$E_2 = [A; 1 - A]$	decreasing
U_3	$D_3 = [\lambda_3; \lambda_4]$	$E_3 = [A; 1 - A]$	increasing
U_4	$D_4 = [\lambda_3; \lambda_4]$	$E_4 = [1 - A; B]$	increasing
U_5	$D_5 = [\lambda_5; \lambda_6]$	$E_5 = [1 - A; B]$	decreasing
U_6	$D_6 = [\lambda_6; \lambda_7]$	$E_6 = [A; 1 - A]$	decreasing
U_7	$D_7 = [\lambda_7; \lambda_8]$	$E_7 = [1 - B; A]$	decreasing
U_8	$D_8 = [\lambda_8; +\infty)$	$E_8 = (0; 1 - B]$	decreasing

Let us construct the set $F_1 := F_1^+ = F_1^-$ of values $(\lambda; \mu)$ such that

$$U(\lambda) + U(\mu) = 1.$$

A couple $(i; j)$ $(i, j = -1, 0, 1, \ldots, 8)$ is called *full*, if $\operatorname{int} E_i + \operatorname{int} E_j \ni 1$, that is,

$$\exists \lambda \in D_i \; \exists \mu \in D_j : \; U_i(\lambda) \in \operatorname{int} E_i, \; U_j(\mu) \in \operatorname{int} E_j, \; U_i(\lambda) + U_j(\mu) = 1.$$

Otherwise $(i; j)$ is called *empty*.

A direct verification shows that there are only 17 full couples (this can be seen from the above table):

$$(0;8),\ (8;0),\ (1;7),\ (7;1),\ (2;2),\ (2;3),\ (3;2),\ (3;3),\ (4;7),$$
$$(7;4),\ (5;7),\ (7;5),\ (6;6),\ (2;6),\ (6;2),\ (3;6),\ (6;3).$$

The rest $10^2 - 17 = 83$ couples are empty ones. For instance, $(-1;-1)$ is not a full couple:

$$\{\operatorname{int} E_{-1}\} + \{\operatorname{int} E_{-1}\} = (2;+\infty) \not\ni 1.$$

On the other hand $(2;3)$ is a full couple:

$$\{\operatorname{int} E_2\} + \{\operatorname{int} E_3\} = \big(2A; 2(1-A)\big) \ni 1,$$

since $0 < A < 0.5$, then $1 - A > 0.5$, and $2A < 1 < 2(1-A)$ follows.

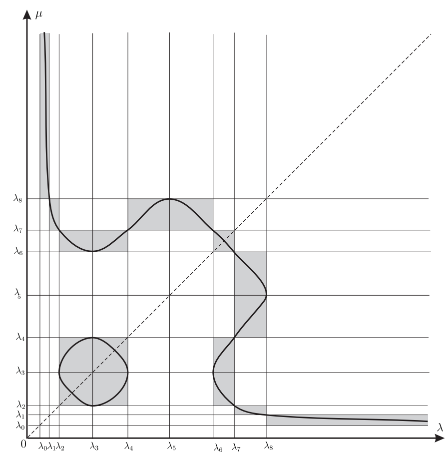


Figure 4. Schematical view of subset $F_1^{\pm}$. The union of $D_i \times D_j$ for full couples $(i;j)$ is shaded.

Due to the conditions of the theorem and the above constructions, one is led to the conclusion that if $(i;j)$ is a *full* couple then

$$\forall \lambda \in D_i \ \ \exists!\, \mu \in D_j : \ \ U_i(\lambda) + U_j(\mu) = 1.$$

Therefore any full couple $(i;j)$ defines the function

$$\mu = U_{ij}(\lambda) = U_i^{-1}\big(1 - U_j(\lambda)\big)$$

that has the domain of definition D_i and the range of values D_j. It follows from the properties of the functions $\mu = U_{ij}(\lambda)$ and location of shaded rectangles (see Fig. 4) that the subset $F_1^{\pm}$ of the spectrum consists of two components, one of them is bounded. □

The cases $A+B = 1$ and $A+B < 1$ can be treated similarly. The respective sets $F_1^{\pm}$ for the function U depicted in Fig. 2 are shown in Fig. 5.

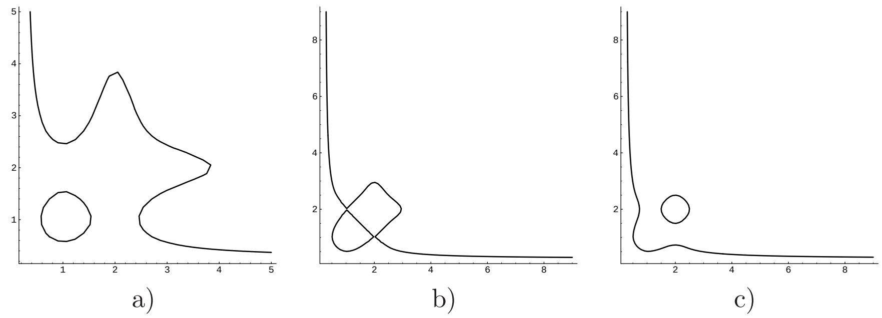


Figure 5. a) $A+B = 0.4+0.75 > 1$, b) $A+B = 0.35+0.65 = 1$, c) $A+B = 0.35+0.55 < 1$.

4 Piece-Wise Linear Functions. Example

Let $f(x)$ be a piece-wise linear function

$$f(x) = \begin{cases} f_1(x), & 0 \le x \le a_1, \\ f_2(x), & a_1 \le x \le a_2, \\ f_3(x), & x \ge a_2, \end{cases}$$

where $f_1(x) = p_1 x + q_1$, $f_2(x) = p_2 x + q_2$, $f_3(x) = p_3 x + q_3$, $f_1(0) = 0$, $f_1(a_1) = f_2(a_1)$, $f_2(a_2) = f_3(a_2)$, $f_3(a_3) = b_3$. Consider equation

$$x'' = -\lambda f(x) + \mu f(-x),$$

where $f(x)$ is a piece-wise linear function depicted in Fig. 6, parameters of the piece-wise linear function $f(x)$ are $a_1 = 0.1$, $a_2 = 0.2$, $a_3 = 0.22$, $b_1 = 0.2$, $b_2 = 0.1$, $b_3 = 120$.

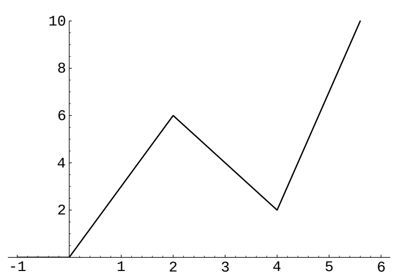
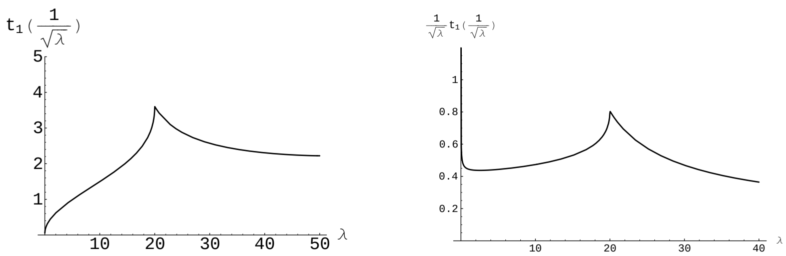


Figure 6. The graphics of function f.

The explicit formula for $t_1(\gamma)$ is given in [1]. So the graphs below are based on precise calculations.

5 Conclusions

The above example shows that the structure of a spectrum may be relatively complicated if the functions f and g in (1.1) are nonlinear. Even for piece-wise linear function f $(g = f)$ the functions $t_1\left(\frac{1}{\sqrt{\lambda}}\right)$ and $\frac{1}{\sqrt{\lambda}} t_1\left(\frac{1}{\sqrt{\lambda}}\right)$ may be



a) b)

Figure 7. a) Function $t_1\left(\frac{1}{\sqrt{\lambda}}\right)$, b) function $\frac{1}{\sqrt{\lambda}}t_1\left(\frac{1}{\sqrt{\lambda}}\right)$, $A+B>1$.

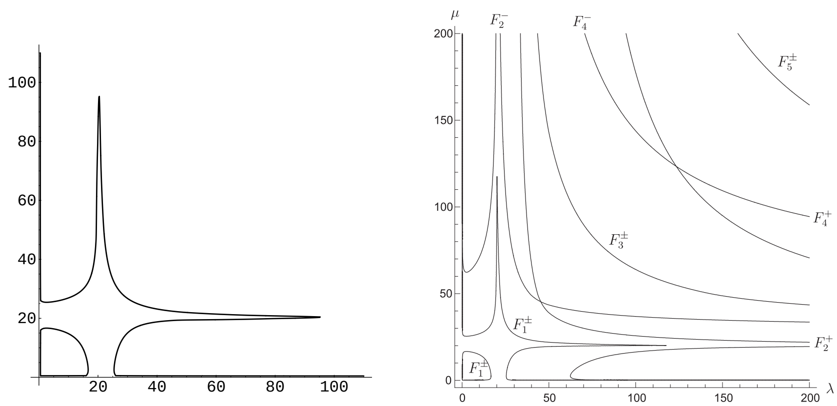


Figure 8. a) The subset $F_1^{\pm}$ consists of two components, the first one being bounded. b) The first five branches of the spectrum.

non-monotone (Fig. 7) and as a result the first branch $F_1^{\pm}$ of the spectrum may consist of two disjoint sets. In Fig. 8 the first branches of the spectrum are depicted. The odd numbered branches coincide $F_1^+ = F_1^-$, $F_3^+ = F_3^-$, $F_5^+ = F_5^-$, the branch $F_1^{\pm}$ consists of two components. The "positive" and "negative" even numbered branches F_2^+ and F_2^-, F_4^+ and F_4^- differ like in the case of the classical Fučik equation. The branch F_2^+ (as well as F_2^-) consists of two infinite components.

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