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A Non-Local Boundary Value Problem for Third-Order Linear Partial Differential Equation of Composite Type

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Abstract. In the paper non-local boundary value problems for a one class of composite type equation with Laplace operator in the main part has been investigated. Using the methods of energy integrals and integral equations, theorems of the uniqueness and existence of a classical solution were proved.

Key words: composite type equation, non-local boundary-value problem, energy integrals, integral equations, Green function, Laplace operator, third order PDE.

1 Introduction

The first investigation of the boundary–value problems for the simple composite type equation of the third order goes back to G. Hadamard [16] and O. Sjöstrand [25], the operator of that equation represents the composition of Laplace operator with an operator of the partial derivative with respect to one of the independent variables. Later many works have appeared such as R. Davis [6], V.V. Daynyak and V.I. Korzyuk [7], T.D. Dzhuraev [9], V.I. Korzyuk and N.I. Yurchuk [18], L. Wolfersdorf [26]. They were devoted to the study of this equation and more general composite type equations. It should be noted that A. Bouziani [2, 3], V.I. Korzyuk and N.I. Yurchuk [18], L. Wolfersdorf [26], give information on the applied aspects of the composite type equations of the third order.

Note that the third order partial differential equations make a base of many mathematical models for various physical and mechanical situations. Many problems associated with the dynamics of the soil moisture and subsoil waters [22], spreading of acoustic waves in a weakly heterogeneous environment [24] are reduced to local and non–local problems for a third order equations. For example, equation

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi(x,y,t)+\beta\frac{\partial}{\partial x}\psi(x,y,t)=f(x,y,t)$$

describes the motion of planetary waves in the β–plane [5], where $\psi(x,y,t)$ is a stream function, defined for the velocity vector $v=(v_1,v_2)$, where $v_1=-\psi_y$, $v_2=\psi_x$, $\beta=2|\Omega_0|cos(\theta/R_0)$, θ is the angle of width place, R_0 is a radius of Earth, Ω_0 is frequency rotation of the fluid, $f(x,y,t)$ is the influence of the forced strength.

The work of T.D. Dzhuraev and Y. Popelek [10] was devoted to the questions of classification and problems of reduction to a canonical form of the third order partial differential equation

$$Au_{xxx}+Bu_{xxy}+Cu_{xyy}+Du_{yyy}=F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}), \tag{1.1}$$

where A,B,C and D are given functions with respects to x and y. If in the every point of the domain the characteristic equation

$$A\lambda^3-B\lambda^2+C\lambda-D=0, \qquad \lambda=\frac{dy}{dx}$$

of the equation (1.1) has one real and two complex conjugate roots, then it can be reduced to the following form

$$\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}\right)(u_{xx}+u_{yy})=F_1(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}). \tag{1.2}$$

In this paper, we will investigate boundary value problems with non–local conditions for the equation (1.2), when F_1 is linear function of u,u_x,u_y, u_{xx}, u_{xy},u_{yy}. Investigation of non–local problems is interesting on theoretical side, because they consist of many local problems. Non–local boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics and etc., see for example [2, 3, 8, 22, 23].

We remark that equation (1.2) often is called the composite type equation. Boundary–value problems for equations of third order with non–local boundary conditions are investigated by A. Bouziani [2, 3], A. Bouziani and M.S. Temi [4], M. Denche and A.L. Marhoune [8], T.D. Dzhuraev and O.S. Zikirov [11, 12], O.S. Zikirov [27] and many references therein.

The present paper is devoted to the study of a non–local problem with the Bitsadze–Samarskiy type conditions for a third order equation of composite type.

2 Statement of the Problem and Uniqueness of the Solution

In the rectangular domain $D=\{(x,y):0<x<p,\ 0<y<q\}$, we consider the third order composite type equation

$$\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}\right)(u_{xx}+u_{yy})+Lu=f(x,y) \tag{2.1}$$

related to one of canonical forms quoted in [10]. Here α and β are some constants, moreover $\alpha^2+\beta^2 \neq 0$, and L be linear differential operator of the second order

$$Lu \equiv a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + a_1(x,y)u_x + b_1(x,y)u_y + c_1(x,y)u.$$

Coefficients and the right side of equation (2.1) are given real–valued functions in the domain D. Without loss of generality, we assume $\alpha \geq 0$, $\beta \geq 0$, but $\alpha^2+\beta^2 \neq 0$.

In the present paper for the equation (2.1) the following non-local conditions are considered.

Problem 1. $S_{\alpha\beta}$. Find a function $u(x,y)$ in the domain D, which is defined for $(x,y)\in D$, continuous with its first derivative in the closed domain $\overline{D}$ and satisfies equation (2.1) in D and also boundary conditions

a) if $\alpha\beta \neq 0$, then the following conditions are posed:

$$u(0,y)=\varphi_1(y), \quad u(p,y)=\varphi_2(y), \qquad 0\leq y\leq q; \tag{2.2}$$

$$u(x,0)=\psi_1(x), \quad u(x,q)=\psi_2(x), \qquad 0\leq x\leq p; \tag{2.3}$$

$$\lambda_1(y)u_x(0,y)+\lambda_2(y)u_x(p,y)=\varphi_3(y), \qquad 0\leq y\leq q; \tag{2.4}$$

$$\mu_1(x)u_y(x,0)+\mu_2(x)u_y(x,q)=\psi_3(x), \qquad 0\leq x\leq p; \tag{2.5}$$

b) if $\beta = 0$, then conditions (2.2)–(2.4) are fulfilled;

c) if $\alpha = 0$, then conditions (2.2), (2.3) and (2.5) are realized, where $\lambda_j(y)$, $\mu_j(x)$, $(j=1,2)$, $\varphi_i(y)$, $\psi_i(x)$, $(i=1,2,3)$ are known functions, moreover

$$\lambda_1^2(y)+\lambda_2^2(y)\neq 0, \quad \mu_1^2(x)+\mu_2^2(x)\neq 0.$$

The data satisfies the following compatibility conditions:

$$\varphi_1(0)=\psi_1(0), \quad \varphi_1(q)=\psi_2(0), \quad \psi_1(p)=\varphi_2(0), \quad \varphi_2(q)=\psi_2(p),$$

$$\lambda_1(0)\psi_1'(0)+\lambda_2(0)\psi_1'(p)=\varphi_3(0), \quad \mu_1(0)\varphi_1'(0)+\mu_2(0)\varphi_1'(q)=\psi_3(0),$$

$$\lambda_1(q)\psi_2'(0)+\lambda_2(q)\psi_1'(p)=\varphi_3(q), \quad \mu_1(p)\varphi_2'(0)+\mu_2(p)\varphi_2'(q)=\psi_3(p).$$

Remark 1. Depending on disposition of the characteristics $\beta x-\alpha y=const$ of the operator $\alpha\dfrac{\partial}{\partial x}+\beta\dfrac{\partial}{\partial y}$ the problem $S_{\alpha\beta}$ includes different local and non–local problems for the equation (2.1).

We state the following designations to further usage. Through $C_{1/2}^{(0,h)}[a,b]$ we designate a set of functions $\varphi(t)$, given in the segment $[a,b]$ and also $[(t-a)(b-t)]^{1/2}\varphi(t)\in C^{(0,h)}[a,b]$, $0<h<1$. If in this set of functions we introduce the norm

$$\|\varphi(t)\|_{h,1/2}=\|\sqrt{(t-a)(b-t)}\varphi(t)\|_{C^h},$$

where $\|\cdot\|_{C^h}$ is a norm in the space $C^{(0,h)}[a,b]$, then the obtained normalized space will be a Banach space [13]. A detailed definition of the spaces $C^{(1,h)}[a,b]$ and $C^k(D)$ can be found, for example in [1].

Definition 1. A function $u(x,y)$ from the class $C^{(1,h)}(\overline{D}) \cap C^3(D)$ is a classical solution of the problem $S_{\alpha\beta}$, if it satisfies equation (2.1) and boundary conditions (2.2)–(2.5).

We can show that for problem $S_{\alpha\beta}$ the case $\alpha\beta < 0$ can be reduced to the case $\alpha\beta > 0$ by changing the independent variable $t = 1 - \tau$. For definiteness we set $\alpha > 0$, $\beta > 0$.

Assumption 2.1 For all $(x,y) \in D$, we assume that $a(x,y)$, $b(x,y)$, $c(x,y) \in C^1(\overline{D}) \cap C^2(D)$, $a_1(x,y), b_1(x,y) \in C(\overline{D}) \cap C^1(D)$, $c_1(x,y) \in C(D)$ and

$$1)\ a(x,y)\xi^2 + 2b(x,y)\xi\eta + c(x,y)\eta^2 \geq c_0(\xi^2+\eta^2),\ \ \forall \xi,\eta \in D,$$

$$2)\ a_{xx} + 2b_{xy} + c_{yy} - a_{1x} - b_{1y} + 2c_1 \leq 0,\ \ \forall (x,y) \in D,$$

$$3)\ \mid \lambda_1(y) \mid > \mid \lambda_2(y) \mid,\quad \mid \mu_1(x) \mid > \mid \mu_2(x) \mid .$$

Assumption 2.2. For all $(x,y) \in D$, we assume $f(x,y) \in C^{(1,h)}(\overline{D})$,

$$\psi_i(x) \in C^{(1,h)}_{1/2}[0,p],\ \ \varphi_i(y) \in C^{(1,h)}_{1/2}[0,q],$$

$$\psi_3(x),\ \mu_i(x) \in C^{(0,h)}_{1/2}[0,p],\ \ \varphi_3(y),\ \lambda_i(y) \in C^{(0,h)}_{1/2}[0,q],\ \ i=1,2,\ 0<h<1.$$

In Assumptions 2.1, 2.2 and in the rest of the paper, we assume that c_j, $j = 0,1,2,\ldots,11$, are positive constants.

In this paper, the existence and uniqueness of a classical solution of the problem $S_{\alpha\beta}$ are demonstrated. We use the methods of energy integrals and integral equations in order to prove the unique solvability of the considered problem.

Theorem 1. *Let Assumption* 2.1 *be fulfilled. Then a classical solution of the problem* $S_{\alpha\beta}$ *is unique.*

Proof. We show that homogeneous problem $S_{\alpha\beta}$, i.e.

$$f(x,y) = 0,\ \ \varphi_i(y) = \psi_i(x) \equiv 0,\ \ i = 1,2,3$$

has only a trivial solution. We prove this fact by using the integral identities. We multiply equation (2.1) by function $u(x,y)$, and integrate the obtained result along the domain D

$$\iint\limits_D u\Big(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\Big)(u_{xx}+u_{yy})\,dxdy + \int\limits_D\int uLu\,dxdy = 0.$$

Obvious identities exist:

$$u\Big(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\Big)(u_{xx}+u_{yy}) = \Big(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\Big)(uu_{xx}+u_{yy})$$
$$-\frac{1}{2}\Big[\Big(\alpha\frac{\partial}{\partial x} - \beta\frac{\partial}{\partial y}\Big)(u_x^2-u_y^2) + \Big(\alpha\frac{\partial}{\partial y} - \beta\frac{\partial}{\partial y}\Big)(2u_xu_y)\Big],$$

$$u\left(a(x,y)u_{xx}+2b(x,y)u_{xy}+c(x,y)u_{yy}\right)=\frac{\partial}{\partial x}\left[auu_x+buu_y-\frac{1}{2}(a_x+b_y)u^2\right]$$
$$+\frac{\partial}{\partial y}\left[buu_x+cuu_y-\frac{1}{2}(b_x+c_y)u^2\right]-(au_x^2+2bu_xu_y+cu_y^2)$$
$$+\frac{1}{2}(a_{xx}+2b_{xy}+c_{yy})u^2,$$
$$u(a_1(x,y)u_x+b_1(x,y)u_y+c_1(x,y)u)=\frac{1}{2}\left[\frac{\partial}{\partial x}(a_1u)^2+\frac{\partial}{\partial y}(b_1u)^2\right]$$
$$-\frac{1}{2}(a_{1x}+b_{1y}-2c_1)u^2.$$

Applying integration by parts to the left part of the last equality and using boundary conditions (2.2)–(2.3), we get

$$\frac{1}{2}\beta\int_0^p\left[u_y^2(x,q)-u_y^2(x,0)\right]dx+\frac{1}{2}\alpha\int_0^q\left[u_x^2(p,y)-u_x^2(0,y)\right]dy$$
$$+\iint_D(au_x^2+2bu_xu_y+cu_y^2)dxdy$$
$$-\frac{1}{2}\iint_D(a_{xx}+2b_{xy}+c_{yy}-a_{1x}-b_{1y}+2c_1)u^2dxdy=0. \tag{2.6}$$

If $\lambda_1(y)\neq 0,\mu_1(x)\neq 0$, then from the conditions (2.4) and (2.5) we find

$$u_x(0,y)=-\frac{\lambda_2(y)}{\lambda_1(y)}u_x(p,y),\quad u_y(x,0)=-\frac{\mu_2(x)}{\mu_1(x)}u_y(x,q).$$

Then (2.6) has a form

$$\frac{1}{2}\beta\int_0^p\left[(1-\frac{\lambda_2(y)}{\lambda_1(y)}\right]u_y^2(x,q)\,dx+\frac{1}{2}\alpha\int_0^q\left[1-\frac{\mu_2(x)}{\mu_1(x)}\right]u_x^2(p,y)\,dy$$
$$+\int_D\int(au_x^2+2bu_xu_y+cu_y^2)\,dxdy$$
$$-\frac{1}{2}\int_D\int(a_{xx}+2b_{xy}+c_{yy}-a_{1x}-b_{1y}+2c_1)u^2\,dxdy=0.$$

It follows from Theorem 1 that in the last expression every item is non–negative. Hence we can conclude $u(x,y)\equiv 0$ in the domain $\overline{D}$. Thus homogeneous equation (2.1) with homogeneous conditions (2.2)–(2.5) has no non–trivial solutions and therefore a solution of the non–local problem $S_{\alpha\beta}$ is unique. □

Remark 2. In the cases $\lambda_j(y)\neq 0$, $\lambda_{3-j}(y)=0$, $\mu_j(x)=0$, $\mu_{3-j}(x)\neq 0$, $j=1,2$ the problem $S_{\alpha\beta}$ is not solvable, because characteristics $\beta x-\alpha y=0$ of the equation (2.1) divide the domain D into two parts, in one of which the given conditions are not sufficient for determining a solution of the problem $S_{\alpha\beta}$ and in the other they are unnecessary.

3 Reducing the Problem $S_{\alpha\beta}$ to the Integral Equation

Existence of the solution of the problem $S_{\alpha\beta}$ for the simplified equation, where $Lu \equiv 0$ and $f(x,y)=0$, has been studied in [11]. Consider the equation

$$\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}\right)(u_{xx}+u_{yy})=g(x,y) \tag{3.1}$$

with homogeneous conditions (2.2)–(2.5). Through $\psi(x)$ we denote an unknown mean of derivative $u_y(x,0)$ if $y=q$, and through $\varphi(y)$ we denote a mean of $u_x(0,y)$ if $x=p$. Let the domain is such that $q \le \beta/\alpha$. Then if we suppose

$$\alpha u_x+\beta u_y=v(x,y), \tag{3.2}$$

then we get from the equation (3.1) that

$$v_{xx}+v_{yy}=g(x,y). \tag{3.3}$$

According to (3.2) and using the boundary conditions (2.2)–(2.5) we get

$$v(0,y)=-\alpha\frac{\lambda_2(y)}{\lambda_1(y)}\varphi(y), \quad v(p,y)=\alpha\varphi(y), \tag{3.4}$$

$$v(x,0)=-\beta\frac{\mu_2(x)}{\mu_1(x)}\psi(x), \quad v(x,q)=\beta\psi(x). \tag{3.5}$$

We shall solve problem (3.3)–(3.5) using the suggestion that $\varphi(y)$ and $\psi(x)$ are continuous and integrable. Moreover,

$$\alpha\frac{\lambda_2(0)}{\lambda_1(0)}\varphi(0)=\beta\frac{\mu_2(0)}{\mu_1(0)}\psi(0), \quad -\alpha\frac{\lambda_2(q)}{\lambda_1(q)}\varphi(q)=\beta\psi(0),$$
$$-\beta\frac{\mu_2(p)}{\mu_1(p)}\psi(p)=\alpha\varphi(0), \qquad \alpha\varphi(q)=\beta\psi(p).$$

On the basis of (3.2), problem (3.1), (2.2)–(2.5) is reduced to the definition of the solution of equation (3.3) with boundary conditions (3.4)–(3.5) in the domain D. The regular solution of the last problem is represented by formula

$$\begin{aligned} v(x,y)=&-\frac{\beta}{2\pi}\int_0^p\Big[\frac{\partial G(x,y;\xi,q)}{\partial\eta}-\frac{\mu_2(\xi)}{\mu_1(\xi)}\frac{\partial G(x,y;\xi,0)}{\partial\eta}\Big]\psi(\xi)\,d\xi\\ &+\frac{\alpha}{2\pi}\alpha\int_0^q\Big[\frac{\partial G(x,y;p,\eta)}{\partial\xi}-\frac{\lambda_2(\eta)}{\lambda_1(\eta)}\frac{\partial G(x,y;0,\eta)}{\partial\xi}\Big]\varphi(\eta)\,d\eta\\ &-\frac{1}{2\pi}\iint_D G(x,y;\xi,\eta)g(\xi,\eta)\,d\xi d\eta, \end{aligned} \tag{3.6}$$

where $G(x,y;\xi,\eta)$ is the Green's function of the Dirichlet problem for the Laplace equation in rectangular [19].

In the domain D we solve the problem

$$\alpha u_x + \beta u_y = v(x,y),$$
$$u(0,y) = 0, \qquad u(x,0) = 0.$$

Its solution can be represented as

$$u(x,y) = \begin{cases} lu_1(x,y), & \text{if} \quad 0 \le \mathrm{x} \le \frac{\alpha}{\beta}\mathrm{y}, \\ u_2(x,y), & \text{if} \quad \frac{\alpha}{\beta}\mathrm{y} \le \mathrm{x} \le \mathrm{p}, \end{cases} \tag{3.7}$$

where

$$u_1(x,y) = \frac{1}{\beta} \int\limits_{y-\beta x/\alpha}^{y} v\Big(x - \frac{\alpha}{\beta}y + \frac{\alpha}{\beta}t, t; \xi, q\Big)\, dt, \qquad 0 \le x \le \frac{\alpha}{\beta}y; \tag{3.8}$$

$$u_2(x,y) = \frac{1}{\beta} \int\limits_{0}^{y} v\Big(x - \frac{\alpha}{\beta}y + \frac{\alpha}{\beta}t, t; \xi, q\Big)\, dt, \qquad \frac{\alpha}{\beta}y \le x \le p; \tag{3.9}$$

Substituting expression (3.7) into (3.8) and (3.9) and changing the order of integration we get

$$\begin{aligned} u_i(x,y) = &- \frac{\beta}{2\pi} \int\limits_0^p K_{1i}(x,y;\xi)\psi(\xi)\, d\xi + \frac{\alpha}{2\pi} \int\limits_0^q K_{2i}(x,y;\eta)\varphi(\eta)\, d\eta \\ &- \frac{1}{2\pi} \iint\limits_D K_i(x,y;\xi,\eta) g(\xi,\eta)\, d\xi d\eta, \quad i = 1,2, \end{aligned} \tag{3.10}$$

where

$$K_{1i}(x,y;\xi) = \int\limits_{z(x,y)}^{y} \Big[\frac{\partial G}{\partial \eta}\Big(x - \frac{\alpha}{\beta}y + \frac{\alpha}{\beta}t, t; \xi, q\Big) - \frac{\mu_2(\xi)}{\mu_1(\xi)} \frac{\partial G}{\partial \eta}\Big(x - \frac{\alpha}{\beta}y + \frac{\alpha}{\beta}t, t; \xi, 0\Big)\Big]\, dt, \tag{3.11a}$$

$$K_{2i}(x,y;\eta) = \int\limits_{z(x,y)}^{y} \Big[\frac{\partial G}{\partial \xi}\Big(x - \frac{\alpha}{\beta}y + \frac{\alpha}{\beta}t, t; p, \eta\Big) - \frac{\lambda_2(\eta)}{\lambda_1(\eta)} \frac{\partial G}{\partial \xi}\Big(x - \frac{\alpha}{\beta}y + \frac{\alpha}{\beta}t, t; 0, \eta\Big)\Big]\, dt, \tag{3.11b}$$

$$K_i(x,y;\xi,\eta) = \int\limits_{z(x,y)}^{y} G\Big(x - \frac{\alpha}{\beta}y + \frac{\alpha}{\beta}t, t; \xi, \eta\Big)\, dt, \ z(x,y) = (2-i)\Big(y - \frac{\beta}{\alpha}x\Big), \ i = 1,2. \tag{3.11c}$$

Relatively to the function (3.10) the following statement is true.

Lemma 1. *If* $g(x,y) \in C^{(1,h)}(\overline{D})$, $\psi(x) \in C^{(0,h)}_{1/2}[0,p]$, $\varphi(y) \in C^{(0,h)}_{1/2}[0,q]$, $0 < h < 1$, *then the function* (3.10) *and its first order derivatives are continuous in the domain* D, *satisfies the equation* (3.1) *and boundary conditions* $u(0,y) = 0$, $u(x,0) = 0$.

Proof. We consider the function $K_{1i}(x,y;\xi)$. We see that

$$\frac{\partial G(x,y;\xi,\eta)}{\partial \eta} = \frac{y-\eta}{(x-\xi)^2+(y-\eta)^2}.$$

Then it follows that

$$K_{1i}(x,y;\xi) = \int\limits_{z(x,y)}^{y} \left[\frac{t-q}{(x-\frac{\alpha}{\beta}y+\frac{\alpha}{\beta}t-\xi)^2+(t-q)^2} + \frac{\mu_2(\xi)}{\mu_1(\xi)} \left(\frac{t}{(x-\frac{\alpha}{\beta}y+\frac{\alpha}{\beta}t-\xi)^2+t^2} \right) \right] dt. \tag{3.11}$$

Calculating the last integral for $i = 1$ we get

$$K_{11}(x,y;\xi) = \frac{\beta^2}{4(\alpha^2+\beta^2)} \Big[\ln|(x-\xi)^2+(y-q)^2| + \frac{\mu_2(\xi)}{\mu_1(\xi)} \ln|(x-\xi)^2+y^2| \Big] + k_{11}(x,y;\xi),$$

where

$$\begin{aligned} k_{11}(x,y;\xi) = &- \frac{\alpha\beta}{2(\alpha^2+\beta^2)} \Big[\arctan \frac{\frac{\alpha}{\beta}(x-\xi)+(y-q)}{(x-\xi)-\frac{\alpha}{\beta}(y-q)} + \frac{\mu_2(\xi)}{\mu_1(\xi)} \arctan \frac{\frac{\alpha}{\beta}(x-\xi)+y}{(x-\xi)-\frac{\alpha}{\beta}y} \Big] \\ &+ \frac{\alpha\beta}{2(\alpha^2+\beta^2)} \Big[\arctan \frac{(y-q)-\frac{\beta}{\alpha}x-\frac{\alpha}{\beta}\xi}{(x-\xi)-\frac{\alpha}{\beta}(y-q)} + \frac{\mu_2(\xi)}{\mu_1(\xi)} \arctan \frac{y-\frac{\beta}{\alpha}x-\frac{\alpha}{\beta}\xi}{(x-\xi)-\frac{\alpha}{\beta}y} \Big] \\ &- \frac{\beta^2}{4(\alpha^2+\beta^2)} \Big[\ln|\xi^2+(y-\frac{\beta}{\alpha}x-q)^2| + \frac{\mu_2(\xi)}{\mu_1(\xi)} \ln|\xi^2+(y-\frac{\beta}{\alpha}x)^2| \Big]. \end{aligned}$$

Calculating (3.11) for $i = 2$ we get

$$K_{12}(x,y;\xi) = \frac{\beta^2}{4(\alpha^2+\beta^2)} \Big[\ln|(x-\xi)^2+(y-q)^2| + \frac{\mu_2(\xi)}{\mu_1(\xi)} \ln|(x-\xi)^2+y^2| \Big] + k_{12}(x,y;\xi),$$

where

$$\begin{aligned} k_{12}(x,y;\xi) = &- \frac{\alpha\beta}{2(\alpha^2+\beta^2)} \Big[\arctan \frac{\frac{\alpha}{\beta}(x-\xi)+(y-q)}{(x-\xi)-\frac{\alpha}{\beta}(y-q)} + \frac{\mu_2(\xi)}{\mu_1(\xi)} \arctan \frac{\frac{\alpha}{\beta}(x-\xi)+y}{(x-\xi)-\frac{\alpha}{\beta}y} \Big] \\ &+ \frac{\alpha\beta}{2(\alpha^2+\beta^2)} \Big[\arctan \frac{\frac{\alpha}{\beta}(x-\xi)-(\frac{\alpha^2}{\beta^2}y+q)}{(x-\xi)-\frac{\alpha}{\beta}(y-q)} + \frac{\mu_2(\xi)}{\mu_1(\xi)} \arctan \frac{\frac{\alpha}{\beta}(x-\frac{\alpha}{\beta}y-\xi)}{(x-\xi)-\frac{\alpha}{\beta}y} \Big] \\ &- \frac{\beta^2}{4(\alpha^2+\beta^2)} \Big[\ln|(x-\frac{\alpha}{\beta}y-\xi)^2+q^2| + \frac{\mu_2(\xi)}{\mu_1(\xi)} \ln|(x-\frac{\alpha}{\beta}y-\xi)^2| \Big]. \end{aligned}$$

Function $k_{1i}(x,y;\xi)$, when $x=\xi$, $y=q$ and $y=0$ is continuous and bounded, and $\dfrac{\partial k_{1i}(x,y;\xi)}{\partial x}$, $\dfrac{\partial k_{1i}(x,y;\xi)}{\partial y}$ are continuous and bounded for all $x\neq\xi$, $y\neq q$ and $y\neq 0$, and for $x\to\xi$, $y\to q$ and $y\to 0$ the estimate takes place

$$\left|\frac{\partial k_{1i}(x,y;\xi)}{\partial x}\right|\le\frac{c_1}{r},\qquad\left|\frac{\partial k_{1i}(x,y;\xi)}{\partial y}\right|\le\frac{c_2}{r},$$

where $r^2=(x-\xi)^2+(y-\eta)^2$. Using the equality

$$\frac{\partial G(x,y;\xi,\eta)}{\partial\xi}=\frac{x-\xi}{(x-\xi)^2+(y-\eta)^2},$$

functions $K_{2i}(x,y;\eta)$ are integrated similarly

$$K_{2i}(x,y;\eta)=\frac{\alpha^2}{4(\alpha^2+\beta^2)}\Big[\ln|(x-p)^2+(y-\eta)^2|+\frac{\lambda_2(\eta)}{\lambda_1(\eta)}\ln|x^2+(y-\eta)^2|\Big]+k_{2i}(x,y;\eta),\quad(3.12)$$

$$\begin{aligned}k_{21}(x,y;\eta)=&\\
&-\frac{\alpha\beta}{2(\alpha^2+\beta^2)}\Big[\arctan\frac{\frac{\alpha}{\beta}(x-p)+(y-\eta)}{(x-p)-\frac{\alpha}{\beta}(y-q)}+\frac{\lambda_2(\eta)}{\lambda_1(\eta)}\arctan\frac{\frac{\alpha}{\beta}x+(y-q)}{x-\frac{\alpha}{\beta}(y-\eta)}\Big]\\
&-\frac{\alpha\beta}{2(\alpha^2+\beta^2)}\Big[\arctan\frac{(y-\eta)-\frac{\beta}{\alpha}x-\frac{\alpha}{\beta}p}{(x-p)-\frac{\alpha}{\beta}(y-\eta)}+\frac{\lambda_2(\eta)}{\lambda_1(\eta)}\arctan\frac{(y-\eta)-\frac{\beta}{\alpha}x}{x-\frac{\alpha}{\beta}(y-\eta)}\Big]\\
&-\frac{\alpha^2}{4(\alpha^2+\beta^2)}\Big[\ln|p^2+(y-\frac{\beta}{\alpha}x-\eta)^2|+\frac{\lambda_2(\eta)}{\lambda_1(\eta)}\ln|(y-\frac{\beta}{\alpha}x-\eta)^2|\Big],\\
k_{22}(x,y;\eta)=&\\
&-\frac{\alpha\beta}{2(\alpha^2+\beta^2)}\Big[\arctan\frac{\frac{\alpha}{\beta}(x-p)+(y-\eta)}{(x-p)-\frac{\alpha}{\beta}(y-q)}+\frac{\lambda_2(\eta)}{\lambda_1(\eta)}\arctan\frac{\frac{\alpha}{\beta}x+(y-q)}{x-\frac{\alpha}{\beta}(y-\eta)}\Big]\\
&-\frac{\alpha\beta}{2(\alpha^2+\beta^2)}\Big[\arctan\frac{\frac{\alpha}{\beta}(x-p)-\frac{\alpha^2}{\beta^2}y-\eta}{(x-p)-\frac{\alpha}{\beta}(y-\eta)}+\frac{\lambda_2(\eta)}{\lambda_1(\eta)}\arctan\frac{\frac{\alpha}{\beta}x-\frac{\alpha^2}{\beta^2}y-\eta}{x-\frac{\alpha}{\beta}(y-\eta)}\Big]\\
&-\frac{\alpha^2}{4(\alpha^2+\beta^2)}\Big[\ln|(x-\frac{\alpha}{\beta}y-p)^2+\eta^2|+\frac{\lambda_2(\eta)}{\lambda_1(\eta)}\ln|(x-\frac{\alpha}{\beta}y)^2+\eta)^2|\Big].\end{aligned}$$

The function $k_{2i}(x,y;\eta)$, when $x=0$, $x=p$ and $y=\eta$, is continuous and bounded, and $\dfrac{\partial k_{1i}(x,y;\xi)}{\partial x}$, $\dfrac{\partial k_{1i}(x,y;\xi)}{\partial y}$ are continuous and bounded for all $x\neq 0$, $x\neq p$ and $y\neq\eta$, and for $x\to 0$, $x\to p$ and $y\to\eta$ the following estimate takes place

$$\left|\frac{\partial k_{2i}(x,y;\eta)}{\partial x}\right|\le\frac{c_3}{r},\qquad\left|\frac{\partial k_{2i}(x,y;\eta)}{\partial y}\right|\le\frac{c_4}{r}.$$

Similarly we prove that functions $K_i(x,y;\xi,\eta)$, $i=1,2$ can be estimated as

$$|K_i(x,y;\xi,\eta)|\le c_5\Big[\frac{\alpha}{\beta}(x-\xi)+(y-\eta)\Big]\ln|(x-\xi)^2+(y-\eta)^2|+c_6,\quad(3.13)$$

and derivatives $K_{ix}(x,y;\xi,\eta)$, $K_{iy}(x,y;\xi,\eta)$ are continuous for all $x\neq\xi$, $y\neq\eta$, and if $x\to\xi$, $y\to\eta$ they have the logarithmic singularity. These facts follow from the expressions (3.11c) and (3.13).

Consequently, the inclusion $u_i(x,y)\in C^{(1,h)}(\overline{D})\bigcap C^3(D)$, $i=1,2$ follows from the theory of harmonic potentials [21] and from the conditions of Lemma 1. If we differentiate (3.10) with respect to x and y we get that $u_i(x,y)$ satisfies equation (3.1). $\square$

We search for a solution of problem $S_{\alpha\beta}$ given in the form of (3.10). Assuming that functions $g(x,y)$, $\psi(x)$, $\varphi(y)$ satisfy the conditions of Lemma 1 we get the function given in (3.10) satisfies all conditions of the problem. To prove this result, we pass to the limit $y\to q$ at $i=1$ and $x\to p$ at $i=2$. Taking into account the continuity of (3.10) we get the following integral equations to determine the functions $\psi(x)$ and $\varphi(y)$:

$$-\frac{\beta}{2\pi}\int\limits_0^p K_{11}(x,q;\xi)\psi(\xi)\,d\xi+\frac{\alpha}{2\pi}\int\limits_0^q K_{21}(x,q;\eta)\varphi(\eta)\,d\eta=\varPhi_1(x),\quad 0\le x\le p, \tag{3.14}$$

$$-\frac{\beta}{2\pi}\int\limits_0^p K_{21}(p,y;\xi)\psi(\xi)\,d\xi+\frac{\alpha}{2\pi}\int\limits_0^q K_{22}(p,y;\eta)\varphi(\eta)\,d\eta=\varPhi_2(y),\quad 0\le y\le q, \tag{3.15}$$

here

$$\varPhi_1(x)=\frac{1}{2\pi}\iint\limits_D K_1(x,q;\xi,\eta)g(\xi,\eta)\,d\xi d\eta,$$
$$\varPhi_2(y)=\frac{1}{2\pi}\iint\limits_D K_2(p,y;\xi,\eta)g(\xi,\eta)\,d\xi d\eta.$$

It is easy to note that the kernels $K_{11}(x,q;\xi)$ and $K_{22}(p,y;\eta)$ of the integral equations (3.14) and (3.15) have a logarithmic singularity at $x=\xi$ and $y=\eta$ respectively, and for them estimates given below are valid. Also, $K_{21}(x,q;\eta)$, $K_{12}(p,y;\xi)$, $\varPhi_1(x)$, $\varPhi_2(y)$ are continuously differentiable functions. This fact follows from Lemma 1 and inequality (3.13).

In the class $C^{1,h}(\overline{D})\bigcap C^3(D)$ the homogeneous problem (3.1), (2.2)–(2.5) with respect to $\psi(x)$ and $\varphi(y)$ is equivalent to the system of integral equations (3.14)–(3.15).

4 Existence of the Solution of Integral Equations

In this section we study question on existence of a solution of the system of the integral equations (3.14)–(3.15) There are various methods for proving the existence of a solution of the first kind Fredholm integral equations. Here we use the proof, which is based on a particular inversion of integral operators [14].

Consider the equation (3.14). By splitting the kernel of the equation into regular and singular parts, we write integral equation (3.14) as

$$-\frac{1}{2\pi}\int_0^p \Big[\ln|x-\xi| + k_{11}(x,\xi)\Big]\psi(\xi)\,d\xi = F(x), \tag{4.1}$$

where

$$k_1(x,\xi) = \frac{\mu_2(\xi)}{\mu_1(\xi)}\ln\left|(x-\xi)^2+q^2\right| + \frac{2\alpha}{\beta}k_{11}(x,q;\xi),$$

$$F(x) = \frac{\alpha^2+\beta^2}{\beta^2}\Phi_1(x) - \frac{\alpha^2}{2\pi\beta^2}\int_0^q K_{21}(x,q;\eta)\varphi(\eta)\,d\eta.$$

From the conditions of Lemma 1 it follows that function $k_1(x,\xi)$ and its first derivatives are continuous and function $F(x)\in C^{(1,h)}[0,p]$.

Lemma 2. *If* $F(x)\in C^{(1,h)}[0,p]$, *then the unique solution* $\psi(x)$ *of the integral equation* (4.1) *exists in the class* $C^{(0,h)}_{1/2}[0,p]$.

Proof. Let us assume that solution $\psi(x)$ of the equation (4.1) exists in the class of functions, which satisfy Hölder's condition in closed interval without endpoints, and around the endpoints this solution can be represented as

$$\psi(x) = \frac{\psi_*(x)}{\sqrt{x(p-x)}},$$

where $\psi_*(x)$ satisfies Hölder's condition. So we represent the equation (4.1) as

$$-\frac{1}{2\pi}\int_0^p \ln|x-\xi|\psi(\xi)d\xi = F_1(x), \tag{4.2}$$

$$F_1(x) = F(x) + \frac{1}{2\pi}\int_0^p k_1(x,\xi)\psi(\xi)\,d\xi.$$

The explicit solution of equation (4.2) is well–known [14, 21] and it can be written by means of the resolvent of the kernel $\ln|x-\xi|$ for $p\neq 4$:

$$\psi(x) = -\frac{1}{4\pi\sqrt{x(p-x)}}\left[\int_0^p \frac{\sqrt{t(p-t)}F_1'(t)\,dt}{t-x} + \frac{1}{\ln(p/4)}\int_0^p \frac{F_1(t)\,dt}{\sqrt{t(p-t)}}\right].$$

According to [17] function $F(x)\in C^{(1,h)}[0,p]$, for $\psi(x)\in C^{(0,h)}[0,p]$. Let us set an unknown function $\psi_*(x)=\psi(x)\sqrt{x(p-x)}$. We get the following integral equation relative to the function $\psi_*(x)$ which is equivalent to the equation (4.1) [15, 17].

$$\psi_*(x) + \int_0^p \frac{M_1(x,\xi)}{\sqrt{\xi(p-\xi)}}\psi_*(\xi)\,d\xi + \int_0^q N_1(x,\eta)\varphi(\eta)\,d\eta = \Phi_3(x), \tag{4.3}$$

where

$$M_1(x,\xi) = \frac{1}{4\pi^2}\Big[\frac{1}{\ln(p/4)}\int\limits_0^p \frac{k_1(t,\xi)dt}{\sqrt{t(p-t)}} - \int\limits_0^p \frac{\sqrt{t(p-t)}k'_{1x}(t,\xi)dt}{t-x}\Big],$$

$$N_1(x,\eta) = \Big(\frac{\alpha}{2\pi\beta}\Big)^3\Big[\frac{1}{\ln(p/4)}\int\limits_0^p \frac{K_{12}(t,q;\eta)dt}{\sqrt{t(p-t)}} - \int\limits_0^p \frac{\sqrt{t(p-t)}K'_{12x}(t,q;\eta)dt}{t-x}\Big],$$

$$\Phi_3(x) = -\frac{1}{\pi^2}\frac{\alpha^2+\beta^2}{\beta^2}\Big[\frac{1}{\ln(p/4)}\int\limits_0^p \frac{\Phi_1(t)dt}{\sqrt{t(p-t)}} - \int\limits_0^p \frac{\sqrt{t(p-t)}\Phi'_1(t)dt}{t-x}\Big].$$

Analogously we inverse the main part of the integral equation (3.15) for $q \neq 4$. We get the equivalent integral equation of the second kind:

$$\varphi_*(x) + \int\limits_0^q \frac{M_2(y,\eta)}{\sqrt{\eta(q-\eta)}}\,\varphi_*(\eta)\,d\eta + \int\limits_0^p N_2(y,\xi)\varphi(\xi)\,d\xi = \Phi_4(y), \qquad (4.4)$$

where $\varphi_*(y) = \sqrt{y(q-y)}\varphi(y)$,

$$M_2(y,\eta) = \frac{1}{4\pi^2}\Big[\frac{1}{\ln(q/4)}\int\limits_0^q \frac{k_2(\tau,\eta)d\tau}{\sqrt{\tau(q-\tau)}} - \int\limits_0^q \frac{\sqrt{\tau(q-\tau)}k'_{2y}(\tau,\eta)d\tau}{\tau-y}\Big],$$

$$N_2(y,\xi) = \Big(\frac{\beta}{2\pi\alpha}\Big)^3\Big[\frac{1}{\ln(q/4)}\int\limits_0^q \frac{K_{21}(p,\tau,\eta)d\tau}{\sqrt{\tau(q-\tau)}} - \int\limits_0^q \frac{\sqrt{\tau(q-\tau)}K'_{21y}(p,\tau,\eta)d\tau}{\tau-y}\Big],$$

$$\Phi_4(y) = -\frac{1}{\pi^2}\frac{\alpha^2+\beta^2}{\alpha^2}\Big[\frac{1}{\ln(q/4)}\int\limits_0^q \frac{\Phi_2(\tau)d\tau}{\sqrt{\tau(q-\tau)}} - \int\limits_0^q \frac{\sqrt{\tau(q-\tau)}\Phi'_2(\tau)d\tau}{\tau-y}\Big].$$

It is shown in [20, 21] that the Fredholm alternative on solvability is applicable to equations (4.3)–(4.4) with kernels $\dfrac{M_1(x,\xi)}{\sqrt{\xi(p-\xi)}}$, $\dfrac{M_2(y,\eta)}{\sqrt{\eta(q-\eta)}}$.

Equations (4.3), (4.4) and problem $S_{\alpha\beta}$ are equivalent, thus their solvability in the class of functions satisfying the Hölder condition follows from the uniqueness Theorem 1.

After defining functions $\psi(x)$ and $\varphi(y)$, we get that a solution of the equation (3.1) satisfying homogeneous boundary conditions (2.2)–(2.5) looks like

$$u(x,y) = \frac{1}{2\pi}\iint\limits_D \mathcal{P}(x,y;\xi,\eta)g(\xi,\eta)\,d\xi d\eta, \qquad (4.5)$$

$$\mathcal{P}(x,y;\xi,\eta) = \frac{1}{\alpha^2+\beta^2}\int_{z(x,y)}^{y}\Big[G_1\Big(x-\frac{\alpha}{\beta}y+\frac{\alpha}{\beta}t,t;\xi,\eta\Big) - \mathcal{S}\Big(x-\frac{\alpha}{\beta}y+\frac{\alpha}{\beta}t,t;\xi,\eta\Big)\Big]\,dt,$$

and $\mathcal{S}(x,y;\xi,\eta)$ is almost defined kernel, depending on the Green function $G_1(x,y;\xi,\eta)$ and its derivatives. This solution is continuous with any order derivatives at $(x,y)\in D$. It is easy to show that function (4.5) at any $g(x,y)\in C^{(1,h)}(\overline{D})$ satisfies equation (3.1) and homogeneous conditions (2.2)–(2.5) [9].

Now we select $g(x,y)$ so that function (4.5) satisfies equation (2.1). Since $g(x,y)\in C^{(1,h)}(\overline{D})$, then derivatives u_x, u_y, u_{xx}, u_{xy}, u_{yy}, $(\Delta u)_x$, and $(\Delta u)_y$ exist, they are continuous in the domain D and

$$\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}\right)(u_{xx}+u_{yy})=2\pi g(x,y).$$

Substituting (4.5) into (2.1) we get the following integral equation

$$g(x,y)=\frac{1}{2\pi}\iint\limits_D \mathcal{K}(x,y;\xi,\eta)g(\xi,\eta)\,d\xi d\eta+f(x,y), \tag{4.6}$$

$$\begin{aligned}\mathcal{K}(x,y;\xi,\eta)=a(x,y)&\frac{\partial^2\mathcal{P}(x,y;\xi,\eta)}{\partial x^2}+2b(x,y)\frac{\partial^2\mathcal{P}(x,y;\xi,\eta)}{\partial xy}\\ &+c(x,y)\frac{\partial^2\mathcal{P}(x,y;\xi,\eta)}{\partial y^2}+a_1(x,y)\frac{\partial\mathcal{P}(x,y;\xi,\eta)}{\partial x}\\ &+b_1(x,y)\frac{\partial\mathcal{P}(x,y;\xi,\eta)}{\partial y}+c_1(x,y)\mathcal{P}(x,y;\xi,\eta).\end{aligned}$$

It is easy to show that function $\mathcal{P}(x,y;\xi,\eta)$ satisfies inequalities

$$\Big|\mathcal{P}_x(x,y;\xi,\eta)\Big|\le c_7\ln|r|,\quad \Big|\mathcal{P}_{xx}(x,y;\xi,\eta)\Big|\le\frac{c_8}{|r|}.$$

We note that the iterated kernel is square integrable. Thus instead of the equation (4.6) we consider integral equation with iterated kernel

$$g(x,y)=\iint\limits_D \mathcal{K}_2(x,y;\xi,\eta)g(\xi,\eta)\,d\xi d\eta+f_1(x,y), \tag{4.7}$$

$$\mathcal{K}_2(x,y;\xi,\eta)=\iint\limits_D \mathcal{K}(x,y;s,t)\mathcal{K}(s,t;\xi,\eta)\,dsdt$$

$$f_1(x,y)=f(x,y)+\iint\limits_D \mathcal{K}(x,y;\xi,\eta)f(\xi,\eta)\,d\xi d\eta.$$

Since $\big|\mathcal{K}(x,y;\xi,\eta)\big|\le c_9/|r|$, then we get [20]:

$$\Big|\mathcal{K}_2(x,y;\xi,\eta)\Big|\le c_{10}\ln r+c_{11}.$$

Consequently the Fredholm theorem is valid for the equation (4.7). Let us note (see, [9]) that the integral equation (4.7) (and problem $S_{\alpha\beta}$) has the unique solution if the following condition takes place

$$\iint\limits_D\iint\limits_D \mathcal{K}_2^2(x,y;\xi,\eta)\,dxdyd\xi d\eta<1.$$

Solving equation (4.7) we find $g(x,y) \in C^{(1,h)}(\bar{D})$ and function $u(x,y)$. It is easy to check that function $u(x,y)$, that is determined by formula (4.5), belongs to the class $g(x,y) \in C^{(1,h)}(\bar{D})$ at $C^{(1,h)}(\bar{D}) \bigcap C^3(D)$. □

Hence, we get the following theorem.

Theorem 2. *Let Assumptions* 2.1 and 2.2 be fulfilled. Then the classical solution of the problem $S_{\alpha\beta}$ exists.

This solution can be represented as (3.10), where $\psi(x)$, $\varphi(y)$ and $g(x,y)$ are already known functions. So existence of the solution of non–local problem $S_{\alpha\beta}$ is proved.

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