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Separable Solutions to an Interacting Human Communities Model

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Abstract. A model for age-structured human communities is studied taking into account a religion factor. The model describes dynamics of interacting religions which tolerate pairs with different confessions, parents can choose a religion not necessarily their own for their offsprings, but it is forbidden to change a confession for any individual. In the case of stationary vital rates, the existence of separable solutions is studied.

Key words: population dynamics, age-structured population, religion factor, human community model.

1 Introduction

Two-sex population models are of great importance for genetics (see e.g. Svirezhev and Passekov [12] and references therein), demography and epidemiology, in particular for modelling sexually transmitted diseases (see e.g. references in Hadeler [2], Prüss and Schappacher [4]). Both random mating (without formation of permanent male-female couples) and monogamous marriage models (see Frederickson [1], Hoppensteadt [3], Staroverov [11], Hadeler [2], Skakauskas [9] and references therein) are usually used. The most general sex-age-structured population deterministic model taking into account marriages has been proposed by Hoppensteadt [3] and Staroverov [11], and consists of a system of three integro-differential equations for the densities of single (unmarried) females, single males, and pairs. Hadeler [2] simplified this model by introducing a maturation period into the mating law.

The Staroverov-Hadeler model has been generalized by Skakauskas [7, 8, 10] taking into account a religion factor and child care. In ecology an individual can be characterized by age and sex. In genetics it can be done by age, sex and a genotype parameter, and in epidemiology age, sex and disease parameters are used. In [7], we extended the list of essential parameters characterizing an individual by adding a religion factor which is very important for pair formation of some human communities. Two different population dynamics models were presented in [7]. The first model tolerates the religion change for the sake of marriage. The second one forbids any confession change of individuals but lets parents to choose a religion not necessarily their own for their newborns.

In the present paper, we consider the second model and study its separable solutions. The plan for this paper is as follows. In Section 2, the model for interacting religion communities is described. In Section 3, conditions for the existence or nonexistence of separable solutions are given. In Section 4, we examine separable solutions in the case of constant vital rates. In Section 5, two lemmas are proved. The concluding remarks are given in Section 6.

2 The Mathematical Model

In this section we describe the model of human communities taking into account interacting religions which forbid any change of the confession for their individuals, but let parents to choose the religion not necessary their own for their newborns [7]. The model consists of the following equations

$$\partial_t u_i^k + \partial_{\tau_k} u_i^k = -\nu_i^k\, u_i^k - L_i^k + S_i^k,\quad t>0,\quad \tau_k \in Q^k,\quad k=1,2, \tag{2.1}$$

$$\partial_t u_{ij}^3 + \sum_{k=1}^{3} \partial_{\tau_k} u_{ij}^3 = -(\nu_{ij}^1 + \nu_{ij}^2 + \sigma_{ij})\, u_{ij}^3,\ t>0,\ (\tau_1,\tau_2,\tau_3)\in Q^3, \tag{2.2}$$

$$L_i^1 = \begin{cases} 0,\ \tau_1<\tau, \\ \displaystyle\sum_{s=1}^{n}\int_{\tau}^{\infty} f_{is}\, d\tau_2,\ \tau_1>\tau, \end{cases} \qquad L_i^2 = \begin{cases} 0, \quad \tau_2<\tau, \\ \displaystyle\sum_{s=1}^{n}\int_{\tau}^{\infty} f_{si}\, d\tau_1,\ \tau_2>\tau, \end{cases} \tag{2.3}$$

$$S_i^1 = \begin{cases} 0, \quad \tau_1<\tau, \\ \displaystyle\int_{0}^{\tau_1-\tau} d\tau_3 \int_{\tau_3+\tau}^{\infty} \sum_{s=i}^{n} (\nu_{is}^2 + \sigma_{is}) u_{is}^3\, d\tau_2,\ \tau_1>\tau, \end{cases} \tag{2.4}$$

$$S_i^2 = \begin{cases} 0, \quad \tau_2<\tau, \\ \displaystyle\int_{0}^{\tau_2-\tau} d\tau_3 \int_{\tau_3+\tau}^{\infty} \sum_{s=i}^{n} (\nu_{si}^1 + \sigma_{si}) u_{si}^3\, d\tau_1,\ \tau_2>\tau \end{cases} \tag{2.5}$$

subject to the conditions

$$u_i^k\Big|_{\tau_1=0} = \int_{0}^{\infty} d\tau_3 \int_{\tau_3+\tau}^{\infty} y\, d\tau_1 \int_{\tau_3+\tau}^{\infty} \sum_{s,j=1}^{n} u_{sj}^3\, b_{sj}^k\, \Omega_{sj,i}\, d\tau_2,\quad t\ge 0,\quad k=1,2, \tag{2.6}$$

$$f_{ij} := u_{ij}^3\Big|_{\tau_3=0} = 2m_{ij} u_i^1 u_j^2 \Big/ \sum_{k=1}^{2}\sum_{s=1}^{n}\int_{\tau}^{\infty} u_s^k\, d\xi,\quad t\ge 0,\quad \tau_1,\tau_2\in(\tau,\infty), \tag{2.7}$$

$$u_i^k\Big|_{t=0} = u_i^{k0},\ \tau_1 \in (0,\infty),\ \ k=1,2, \quad u_{ij}^3\Big|_{t=0} = u_{ij}^{30},\ \ (\tau_1,\tau_2,\tau_3) \in Q^3, \quad (2.8)$$

$$[u_i^k\big|_{\tau_k=\tau}] = 0,\ \ t>0,\ \ k=1,2, \quad \sum_{s=1}^{n} \Omega_{ij,s} = 1,\ \ t>0,\ \ (\tau_1,\tau_2,\tau_3) \in \bar{Q}^3.$$

Here $i,j=1,\dots,n$, where n is the number of interacting religions, $Q^k = (0,\tau)\cup(\tau,\infty)$, $k=1,2$, $Q^3 = \{(\tau_1,\tau_2,\tau_3) : \tau_1 \in (\tau_3+\tau,\infty),\ \tau_2 \in (\tau_3+\tau,\infty),\ \tau_3 \in (0,\infty)\}$, $u_i^1(t,\tau_1)$ is the density at time t of single (unmarried) males of age τ_1 and of the ith religion, $u_i^2(t,\tau_2)$ is the density at time t of single (unmarried) females of age τ_2 and of the ith religion, $u_{ij}^3(t,\tau_1,\tau_2,\tau_3)$ is the density at time t of a pair which is formed of a male of age τ_1 and of the ith religion and a female of age τ_2 and of the jth religion, and which have existed for τ_3 units of time, $\nu_i^1(t,\tau_1)$ denotes the death rate at time t of single males of age τ_1 and of the ith religion, $\nu_i^2(t,\tau_2)$ is the death rate at time t of single females of age τ_2 and of the ith religion; $\nu_{ij}^1(t,\tau_1,\tau_2,\tau_3)$ denotes the death rate at time t of males from a pair formed of a male of age τ_1 and of the ith religion and a female of age τ_2 and of the jth religion, and which have existed for τ_3 units of time, $\nu_{ij}^2(t,\tau_1,\tau_2,\tau_3)$ is the death rate at time t of females from a pair formed of a male of age τ_1 and of the ith religion and a female of age τ_2 and of the jth religion, and which have existed for τ_3 units of time, $\sigma_{ij}(t,\tau_1,\tau_2,\tau_3)$ is the divorce rate at time t of pairs formed of a male of age τ_1 and of the ith religion and a female of age τ_2 and of the jth religion, and which have existed for τ_3 units of time, $b_{ij}^1(t,\tau_1,\tau_2,\tau_3)$ denotes the birth rate at time t of males produced by a pair formed of a male of age τ_1 and of the ith religion and a female of age τ_2 and of the jth religion, and which have existed for τ_3 units of time, $b_{ij}^2(t,\tau_1,\tau_2,\tau_3)$ is the birth rate at time t of females produced by a pair formed of a male of age τ_1 and of the ith religion and a female of age τ_2 and of the jth religion, and which have existed for τ_3 units of time, $L_i^k(t,\tau_k)$, $k=1,2$ are the single individuals of age τ_k and of the ith religion loss rate caused by the marriage, $S_i^k(t,\tau_k)$, $k=1,2$ are the single individuals of age τ_1 and of the ith religion gain rate caused by the death of the pair partner and divorce of pairs, $f_{ij}(t,\tau_1,\tau_2)$ denotes the formation rate of pairs consisting of a male of age τ_1 and of the ith religion and a female of τ_2 age and of the jth religion, $\Omega_{sj,i}(t,\tau_1,\tau_2,\tau_3)$ is the probability that a pair, formed of a male of age τ_1 and of the ith religion and a female of age τ_2 and of the jth religion and which have existed for τ_3 units of time, will choose the ith religion for its newborn produced at time t, $u_i^{k0}(\tau_k)$, $k=1,2$; $u_{ij}^{30}(\tau_1,\tau_2,\tau_3)$ are the initial distributions, $[u^k(t,\tau)]$ denotes the jump discontinuity of u_i^k at the line $\tau_k=\tau$, $k=1,2$.

3 Separable Solutions

In this section, we examine system (2.1)–(2.8) with vital rates ν_i^k, ν_{ij}^k, σ_{ij}, b_i^k, m_{ij}, and $\Omega_{sj,i}$ independent of t and look for solutions of the form

$$\begin{cases} u_i^k(t,\tau_k) = U_i^k(\tau_k)\exp\{\lambda t\},\ u_i^{k0} = U_i^k, \\ u_{ij}^3(t,\tau_1,\tau_2,\tau_3) = U_{ij}^3(\tau_1,\tau_2,\tau_3)\exp\{\lambda t\},\ u_{ij}^{30} = U_{ij}^3, \end{cases} \quad (3.1)$$

where U_i^k, U_{ij}^3, and the constant λ are to be determined. We substitute Eq. (3.1) into (2.1)–(2.7) to obtain

$$\begin{cases} \dfrac{dU_i^k}{d\tau_k} = -(\lambda + \nu_i^k)U_i^k, \ \tau_k \in (0,\tau], \\ U_i^k(0) = \int\limits_0^\infty d\tau_3 \int\limits_{\tau_3+\tau}^\infty d\tau_1 \int\limits_{\tau_3+\tau}^\infty \sum\limits_{s,j=1}^n U_{sj}^3\, b_{sj}^k\, \Omega_{sj,i}\, d\tau_2, \ k = 1,2, \end{cases} \tag{3.2}$$

$$\frac{dU_i^1}{d\tau_1} = -\Big(\lambda + \nu_i^1 + \frac{2}{f}\sum_{s=1}^n \int\limits_\tau^\infty m_{is} U_s^2\, d\tau_2\Big)U_i^1 + \int\limits_0^{\tau_1-\tau} d\tau_3 \int\limits_{\tau_3+\tau}^\infty \sum_{s=i}^n (\nu_{is}^2 + \sigma_{is}) U_{is}^3 d\tau_2,$$

$$\tau_1 > \tau, \ [U_i^1(\tau)] = 0, \tag{3.3}$$

$$\frac{dU_i^2}{d\tau_2} = -\Big(\lambda + \nu_i^2 + \frac{2}{f}\sum_{s=1}^n \int\limits_\tau^\infty m_{si} U_s^1 d\tau_1\Big)U_i^2 + \int\limits_0^{\tau_2-\tau} d\tau_3 \int\limits_{\tau_3+\tau}^\infty \sum_{s=i}^n (\nu_{si}^1 + \sigma_{si}) U_{si}^3 d\tau_2,$$

$$\tau_2 > \tau, \ [U_i^2(\tau)] = 0, \tag{3.4}$$

$$\begin{cases} \sum\limits_{\alpha=1}^3 \partial_{\tau_k} U_{ij}^3 = -(\lambda + \nu_{ij}^1 + \nu_{ij}^2 + \sigma_{ij})\, U_{ij}^3, \ \ (\tau_1,\tau_2,\tau_3) \in Q^3, \\ U_{ij}^3\Big|_{\tau_1=0} = \dfrac{2m_{ij}U_i^1U_j^2}{f}, \end{cases} \tag{3.5}$$

where $f = \sum\limits_{k=1}^2 \sum\limits_{\alpha=1}^n \int\limits_\tau^\infty U_\alpha^k(\tau_k)\, d\tau_k$. Set

$$U_i^k = f w_i^k, \ U_{ij}^3 = f w_{ij}^3. \tag{3.6}$$

Then

$$\sum_{k=1}^2 \sum_{\alpha=1}^n \int\limits_\tau^\infty w_\alpha^k(\tau_k)\, d\tau_k = 1. \tag{3.7}$$

Formal integration of Eq. (3.5) gives

$$w_{ij}^3(\tau_1,\tau_2,\tau_3) = w_i^1(\tau_1-\tau_3) w_j^2(\tau_2-\tau_3) K_{ij}(\tau_1-\tau_3, \tau_2-\tau_3, \tau_3; \lambda), \tag{3.8}$$

where

$$\begin{aligned} &K_{ij}(\tau_1-\tau_3,\tau_2-\tau_3,\tau_3;\lambda) \\ &\quad = 2m_{ij}(\tau_1-\tau_3,\tau_2-\tau_3)\exp\Big\{-\int\limits_0^{\tau_3}\big(\tilde\nu_{ij}(\xi+\tau_1-\tau_3,\xi+\tau_2-\tau_3,\xi)+\lambda\big)d\xi\Big\} \\ &\quad = K_{ij}(\tau_1-\tau_3,\tau_2-\tau_3,\tau_3;0)\exp\big(-\lambda\tau_3\big), \ \ \tilde\nu_{ij} = \nu_{ij}^1 + \nu_{ij}^2 + \sigma_{ij}. \end{aligned}$$

Inserting (3.6) and (3.8) into (3.3), (3.4), and $((3.2))_2$ we get

$$\frac{d\,w_i^1}{d\tau_1} = -\Big(\lambda + \nu_i^1 + 2\sum_{s=1}^{n}\int_{\tau}^{\infty} m_{is} w_s^2\, d\tau_2\Big) w_i^1 + \int_{0}^{\tau_1-\tau} d\tau_3 \int_{\tau_3+\tau}^{\infty} \sum_{s=i}^{n} (\nu_{is}^2 + \sigma_{is}) w_i^1$$
$$\times (\tau_1-\tau_3) w_s^2(\tau_2-\tau_3) K_{is}(\tau_1-\tau_3, \tau_2-\tau_3, \tau_3; \lambda)\, d\tau_2, \quad \tau_1 > \tau,\ [w_i^1(\tau)] = 0, \tag{3.9}$$

$$\frac{d\,w_i^2}{d\tau_2} = -\Big(\lambda + \nu_i^2 + 2\sum_{s=1}^{n}\int_{\tau}^{\infty} m_{si} w_s^1\, d\tau_1\Big) w_i^2 + \int_{0}^{\tau_2-\tau} d\tau_3 \int_{\tau_3+\tau}^{\infty} \sum_{s=i}^{n} (\nu_{si}^1 + \sigma_{si}) w_s^1$$
$$\times (\tau_1-\tau_3) w_i^2(\tau_2-\tau_3) K_{si}(\tau_1-\tau_3, \tau_2-\tau_3, \tau_3; \lambda)\, d\tau_1, \quad \tau_2 > \tau,\ [w_i^2(\tau)] = 0, \tag{3.10}$$

$$w_i^k(0) = \int_{0}^{\infty} d\tau_3 \int_{\tau_3+\tau}^{\infty} d\tau_1 \int_{\tau_3+\tau}^{\infty} \sum_{s,j=1}^{n} (b_{sj}^k \varOmega_{sj,i})\big|_{(\tau_1,\tau_2,\tau_3)} w_s^1(\tau_1-\tau_3) w_j^2(\tau_2-\tau_3)$$
$$\times K_{sj}(\tau_1-\tau_3, \tau_2-\tau_3, \tau_3; \lambda)\, d\tau_1, \quad k = 1, 2, \tag{3.11}$$

Set $\bar{w}^1 = (w_1^1, \ldots, w_n^1)$, $\bar{w}^2 = (w_1^2, \ldots, w_n^2)$,

$$r_i^1(\tau_1, \bar{w}^2) = 2\sum_{s=1}^{n}\int_{\tau}^{\infty} m_{is}(\tau_1,\tau_2) w_s^2(\tau_2) d\tau_2,$$

$$r_i^2(\tau_2, \bar{w}^1) = 2\sum_{s=1}^{n}\int_{\tau}^{\infty} m_{si}(\tau_1,\tau_2) w_s^1(\tau_1) d\tau_1,$$

$$R_i^1(\tau_1, x; \bar{w}^2, \lambda) = \int_{\tau}^{\infty} \sum_{s=1}^{n} w_s^2(y) K_{is}(x, y, \tau_1 - x; \lambda)$$
$$\times (\nu_{is}^2 + \sigma_{is})\big|_{(\tau_1, y+\tau_1-x, \tau_1-x)} dy = R_i^1(\tau_1, x; \bar{w}^2, 0) \exp\{-\lambda(\tau_1 - x)\},$$

$$R_i^2(x, \tau_2; \bar{w}^1, \lambda) = \int_{\tau}^{\infty} \sum_{s=1}^{n} w_s^1(y) K_{si}(y, x, \tau_2 - x; \lambda)$$
$$\times (\nu_{si}^1 + \sigma_{si})\big|_{(y+\tau_2-x, \tau_2, \tau_2-x)} dy = R_i^2(x, \tau_2; \bar{w}^1, 0) \exp\{-\lambda(\tau_2 - x)\},$$

$$\kappa_{sj,i}^k(x, y; \lambda) = \int_{0}^{\infty} K_{sj}(x, y, \tau_3; \lambda)(b_{sj}^k \varOmega_{sj,i})\big|_{(x+\tau_3, y+\tau_3, \tau_3)} d\tau_3.$$

This allows us to rewrite Eqs. (3.9)–(3.11) in the form

$$\frac{d\,w_i^1}{d\tau_1}=-\big(\lambda+\nu_i^1+r_i^1(\tau_1;\bar{w}^2)\big)w_i^1+\int_\tau^{\tau_1} w_i^1(x)R_i^1(\tau_1,x;\bar{w}^2,\lambda)\,dx,$$
$$w_i^1\big|_{\tau_1=\tau}=w_i^1(\tau), \tag{3.12}$$
$$\frac{d\,w_i^2}{d\tau_1}=-\big(\lambda+\nu_i^2+r_i^2(\tau_2;\bar{w}^1)\big)w_i^2+\int_\tau^{\tau_2} w_i^2(x)R_i^2(x,\tau_2;\bar{w}^1,\lambda)dx,$$
$$w_i^2\big|_{\tau_2=\tau}=w_i^2(\tau), \tag{3.13}$$
$$w_i^k(0)=\int_\tau^\infty dx\int_\tau^\infty \sum_{s,j=1}^n w_s^1(x)w_j^2(y)\kappa_{sj,i}^k(x,y;\lambda)dy, \quad k=1,2. \tag{3.14}$$

From Eqs. (3.2) and $(3.6)_1$ it follows that

$$w_i^k(\tau_k)=w_i^k(0)f_i^k(\tau_k;\lambda), \quad f_i^k(\tau_k;\lambda)=\exp\Big\{-\int_0^{\tau_k}(\lambda+\nu_i^k(x))dx\Big\}.$$

Hence,

$$w_i^k(\tau)=w_i^k(0)f_i^k(\tau;\lambda). \tag{3.15}$$

It remains to solve Eqs. (3.12)–(3.15) and (3.7). Therefore, separable solutions (3.1) of Eqs. (2.1)–(2.7) correspond to solutions $(\bar{w}^1,\bar{w}^2,\lambda)$ of problem (3.12)–(3.15) and (3.7). To treat the existence or nonexistence of separable solutions we follow the papers [10, 9] in which we used the Prüss and Schappacher [4] method applied for the investigation of separable solutions to the Staroverov [11] model with the Keyfitz-Hadeler pair formation function [5]. According to this method, we have to reformulate system (3.12)–(3.15) and then apply Schauder's fixed-point principle [6]. For this reformulation we choose the space

$$X=\Big\{(\bar{w}^1,\bar{w}^2): w_s^k\in L_+^1(\tau,\infty), \quad k=1,2; \quad s=1,\dots,n\Big\}$$

and set

$$D=\Big\{(\bar{w}^1,\bar{w}^2):(\bar{w}^1,\bar{w}^2)\in X, \quad \sum_{k=1}^2\sum_{s=1}^n \| w_s^k \|=1\Big\},$$

where $\|\cdot\|:=\|\cdot\|_{L^1(\tau,\infty)}$. Obviously, D is closed, bounded, and convex. We shall construct the operator $F^\lambda: D\to D$ with at least one fixed point in D. To do this, we linearize system (3.12) and (3.13) by prescribing $(\bar{w}^1,\bar{w}^2)\in D$ involved in $r_i^1(\tau_1;\bar{w}^2), r_i^2(\tau_2;\bar{w}^1)$, $R_i^1(\tau_1,x;\bar{w}^2,\lambda)$, and $R_i^2(\tau_2,x;\bar{w}^1,\lambda)$, $i=1,\dots,n$. Then, for given $(\bar{w}^1,\bar{w}^2)$ and a real λ, we define

$$l_i^1(\tau_1;\lambda)=\lambda+\nu_i^1(\tau_1)+r_i^1(\tau_1;\bar{w}^2), \quad l_i^2(\tau_2;\lambda)=\lambda+\nu_i^2(\tau_2)+r_i^2(\tau_2;\bar{w}^1),$$
$$g_i^1(\tau_1,x;\lambda)=R_i^1(\tau_1,x;\bar{w}^2,\lambda), \quad g_i^2(x,\tau_2;\lambda)=R_i^2(x,\tau_2;\bar{w}^1,\lambda)$$

with $i = 1, \dots, n$ and rewrite Eqs. (3.12) and (3.13) as follows:

$$\frac{d\,w_i^1}{d\tau_1} = -l_i^1(\tau_1;\lambda)w_i^1 + \int_{\tau}^{\tau_1} w_i^1(x)g_i^1(\tau_1,x;\lambda)dx, \;\; w_i^1(\tau) = w_i^1(0)f_i^1(\tau;\lambda), \quad (3.16)$$

$$\frac{d\,w_i^2}{d\tau_2} = -l_i^2(\tau_2;\lambda)w_i^2 + \int_{\tau}^{\tau_1} w_i^2(x)g_i^2(x,\tau_2;\lambda)dx, \;\; w_i^2(\tau) = w_i^2(0)f_i^2(\tau;\lambda) \quad (3.17)$$

with $w_i^k(0)$ defined by Eq. (3.14). Letting

$$w_i^k(\tau_k) = w_i^k(\tau)z_i^k(\tau_k) \quad (3.18)$$

with $k = 1, 2$, $i = 1, \dots, n$, and $w_i^k(\tau)$ defined by (3.15) from Eqs. (3.16) and (3.17) we get

$$\frac{d\,z_i^1}{d\tau_1} = -l_i^1(\tau_1;\lambda)z_i^1 + \int_{\tau}^{\tau_1} z_i^1(x)g_i^1(\tau_1,x;\lambda)\,dx, \;\; z_i^1(\tau) = 1, \quad (3.19)$$

$$\frac{d\,z_i^2}{d\tau_2} = -l_i^2(\tau_2;\lambda)z_i^2 + \int_{\tau}^{\tau_1} z_i^2(x)g_i^2(x,\tau_2;\lambda)\,dx, \;\; z_i^2(\tau) = 1. \quad (3.20)$$

Note that these equations can be examined separately and that

$$z_i^1 = z_i^1(\tau_1;\bar{w}^2,\lambda) = z_i^1(\tau_1;\bar{w}^2,0)\exp\big\{-\lambda(\tau_1-\tau)\big\},$$
$$z_i^2 = z_i^2(\tau_2;\bar{w}^1,\lambda) = z_i^2(\tau_2;\bar{w}^1,0)\exp\big\{-\lambda(\tau_2-\tau)\big\}.$$

By formal integration of Eqs. (3.19) and (3.20) we derive

$$z_i^1(\tau_1) = \exp\Big\{-\int_{\tau}^{\tau_1} l_i^1(\xi;\lambda)d\xi\Big\} + \int_{\tau}^{\tau_1} \exp\Big\{-\int_{\eta}^{\tau_1} l_i^1(\xi;\lambda)d\xi\Big\}\Big(\int_{\tau}^{\eta} z_i^1(x)g_i^1(\eta,x;\lambda)dx\Big)d\eta, \quad (3.21)$$

$$z_i^2(\tau_2) = \exp\Big\{-\int_{\tau}^{\tau_2} l_i^2(\xi;\lambda)d\xi\Big\} + \int_{\tau}^{\tau_2} \exp\Big\{-\int_{\eta}^{\tau_2} l_i^2(\xi;\lambda)d\xi\Big\}\Big(\int_{\tau}^{\eta} z_i^2(x)g_i^2(x,\eta;\lambda)\,dx\Big)d\eta, \quad (3.22)$$

which can be written in the form

$$z_i^1(\tau_1) = \exp\Big\{-\int_{\tau}^{\tau_1} l_i^1(\xi;\lambda)d\xi\Big\} + \int_{\tau}^{\tau_1} z_i^1(x)G_i^1(\tau_1,x;\lambda)\,dx, \quad (3.23)$$

$$z_i^2(\tau_2) = \exp\Big\{ -\int_\tau^{\tau_2} l_i^2(\xi;\lambda)d\xi\Big\} + \int_\tau^{\tau_2} z_i^2(x) G_i^2(x,\tau_2;\lambda)\,dx \tag{3.24}$$

with

$$G_i^1(\tau_1,x;\lambda) = \int_x^{\tau_1} g_i^1(\eta,x;\lambda)\exp\Big\{ -\int_\eta^{\tau_1} l_i^1(\xi;\lambda)d\xi\Big\}d\eta,$$

$$G_i^2(x,\tau_2;\lambda) = \int_x^{\tau_2} g_i^2(x,\eta;\lambda)\exp\Big\{ -\int_\eta^{\tau_2} l_i^2(\xi;\lambda)d\xi\Big\}d\eta.$$

Set $\bar z^k = (z_1^k,\dots,z_n^k)$,

$m^* = \max\limits_{i,s} \sup\limits_{[\tau,\infty)^2} m_{is}, \quad \nu_* = \min\big(\min\limits_{k,i}\inf\limits_{Q^k}\nu_i^k,\ \min\limits_{k,i,s}\inf\limits_{Q^3}\nu_{is}^k\big),$

$$z_{i*}^k(\tau_k,\lambda) = \exp\Big\{-\int_\tau^{\tau_k}(\lambda+\nu_i^k(x)+2m^*)dx\Big\},\ z^{k*}(\tau_k,\lambda) = \exp\Big\{-(\lambda+\nu_*)(\tau_k-\tau)\Big\},$$

with $\lambda > -\nu_*$,

$$\tilde D = \Big\{(\bar z^1,\bar z^2) : z_i^k \in C^1([\tau,\infty)), z_{i*}^k \le z_i^k \le z^{k*}, \Big\|\frac{dz_i^k}{d\tau_k}\Big\| \le 1 + \frac{4m^*}{\lambda+\nu_*}, k=1,2\Big\}.$$

Lemma 1. *Let ν_* and σ_{is} be positive, m_{is} be positive and bounded, and let $\nu_i^k \in C^0([\tau,\infty))$, $m_{is} \in C^0([\tau,\infty)^2)$, ν_{ij}^k and $\sigma_{ij} \in C^0(\bar Q^3)$, $(\bar w^1,\bar w^2) \in D$. Then, for a real $\lambda > -\nu_*$, the following assertions are true:*

(i) *Eqs.* (3.23) *and* (3.24) *have a unique positive solution $z_i^1(\tau_1; w^2,\lambda)$ and $z_i^2(\tau_2; w^1,\lambda)$ such that*

$$z_i^k \in C^0\big([\tau,\infty)) \cap C^1(\tau,\infty)\big), \quad z_{i*}^k \le z_i^k \le z_i^{k*}, \quad \Big\|\frac{dz_i^k}{d\tau_k}\Big\| \le 1 + \frac{4m^*}{\lambda+\nu_*},$$

that is $(\bar z^1,\bar z^2) \in \tilde D$.

(ii) *The operator $\tilde F^\lambda : (\bar w^1,\bar w^2) \to \big(\bar z^1(\cdot,\bar w^2,\lambda), \bar z^2(\cdot,\bar w^1,\lambda)\big)$ from D to X is completely continuous (compact), moreover $\tilde F^\lambda(D) \subset \tilde D \subset X$.*

The proof of Lemma 1 is given in Section 4.

Inserting (3.18) with $w_i^k(\tau)$ defined by (3.15) into (3.14), we get the system

$$w_i^k(0) = \sum_{s,j=1}^n p_{sj,i}^k(\bar w^1,\bar w^2,\lambda) w_s^1(0) w_j^2(0) \tag{3.25}$$

with

$$p_{sj,i}^k(\bar w^1,\bar w^2,\lambda) = f_s^1(\tau;\lambda) f_j^2(\tau;\lambda) \times \int_\tau^\infty dx \int_\tau^\infty \kappa_{sj,i}^k(x,y;\lambda)\, z_s^1(x,\bar w^2,\lambda)\, z_j^2(y;\bar w^1,\lambda)\,dy.$$

Examination of the solution to system (3.25) and its behaviour as $\lambda \to \infty$ in general case of vital rates is a complicated problem. Therefore, we restrict ourselves to the consideration of the following two cases:

1. The vital functions m_{is}, ν_{is}^k, ν_i^k, and σ_{is} satisfy conditions of Lemma 1 and do not depend on the confession number, functions ν_{is}^k and ν_i^k increase as ages tend to ∞, $b_{ij}^k = \beta_{ij}^k b(\tau_1, \tau_2, \tau_3)$ with positive constants β_{ij}^k and positive bounded $b \in C^0(\bar{Q}^3)$, all $\Omega_{sj,i}$ are nonnegative constants;

2. All vital rates are continuous and bounded with positive lower bounds, $n = 2$, and sex ratio of newborns is a constant, i.e., $b_{ij}^2 = \gamma b_{ij}^1$.

Case 1. Definition of f_i^k, r_i^k, l_i^k, R_i^k, K_{ij}, and G_i^k and Eqs. (3.23) and (3.24) show that z_i^k is also independent of the confession number. Letting $\nu^k = \nu_i^k$, $\tilde{\nu}^k = \nu_{ij}^k$, $\sigma = \sigma_{ij}$, $z^k = z_i^k$, $K = K_{ij}$, and $f^k(\tau, \lambda) = f_i^k(\tau, \lambda)$ we rewrite Eqs. (3.25) as follows

$$w_i^k(0) = \sum_{s,j=1}^{n} w_s^1(0) w_j^2(0) a_{sj,i}^k q(\bar{w}^1, \bar{w}^2, \lambda), \;\; a_{sj,i}^k = \beta_{sj}^k \Omega_{sj,i}, \tag{3.26}$$

$$q(\bar{w}^1, \bar{w}^2, \lambda) = f^1(\tau, \lambda) f^2(\tau, \lambda) \int_{\tau}^{\infty} z^1(x, \bar{w}^2, \lambda) dx \int_{0}^{\infty} z^2(x, \bar{w}^1, \lambda) dy$$

$$\times \int_{0}^{\infty} \exp\{-\lambda \tau_3\} K(x + \tau_3, y + \tau_3, \tau_3, 0) b(x + \tau_3, y + \tau_3, \tau_3) d\tau_3 \tag{3.27}$$

We examine system (3.26) in two cases: (1_1) all $\Omega_{sj,i} > 0$ and (1_2) all $\Omega_{sj,i} \geq 0$. In the first case the following statement is true.

Lemma 2. *Let $\Omega_{sj,i}$ and β_{ij}^k be positive constants and $b \in C^0(\bar{Q}^3)$ be positive and bounded. Then under the conditions of Lemma 1 system* (3.26) *has a positive solution* $(\bar{w}^1(0), \bar{w}^2(0))$ *with*

$$w_i^k(0) = \frac{h_i^k}{q(\bar{w}^1, \bar{w}^2, \lambda)}, \quad h_i^k = \phi_i^k / \sum_{s,j=1}^{n} \phi_s^1 \phi_j^2 a_{sj,1}^1 \quad i = 1, \ldots, n, \tag{3.28}$$

where $\phi_1^1 = 1$, while $(\phi_2^1, \ldots, \phi_n^1, \phi_1^2, \ldots, \phi_n^2)$ is at least one positive solution of the system

$$\phi_i^k = \sum_{s,j=1}^{n} \phi_s^1 \phi_j^2 a_{sj,i}^k \Big/ \sum_{s,j=1}^{n} \phi_s^1 \phi_j^2 a_{sj,1}^1, \quad i = 2, \ldots, n.$$

The proof of Lemma 2 is given Section 5.

Next we consider the case (1_2). More precisely, we examine a usual case for some confessions, where

$$\begin{aligned} &\Omega_{ss,s} = 1, \\ &\Omega_{sj,k} = 0, \quad s, j \neq k, \\ &\Omega_{sj,s} + \Omega_{sj,j} = 1, \;\; \Omega_{sj,s} > 0, \;\; \Omega_{sj,j} > 0, \;\; s \neq j. \end{aligned} \tag{3.29}$$

In this case, system (3.26) reduces to

$$\frac{w_i^k(0)}{q(\bar w^1,\bar w^2,\lambda)} = w_i^1(0)w_i^2(0)a_{ii,i}^k + \sum_{j\neq i}\bigl(w_i^1(0)w_j^2(0)a_{ij,i}^k + w_j^1(0)w_i^2(0)a_{ji,i}^k\bigr). \tag{3.30}$$

We examine this system only in the case where the sex ratio of newborns is constant, i.e., $\beta_{ij}^2 = \gamma\beta_{ij}^1$. It is easy to see that

$$w_i^k(0) = \frac{h_i^k}{q(\bar w^1,\bar w^2,\lambda)}, \quad h_i^1 = \frac{\tilde w_i}{\gamma}, \quad h_i^2 = \tilde w_i, \quad i = 1,2,\dots,n, \tag{3.31}$$

where $(\tilde w_1,\dots,\tilde w_n)$ is a positive solution of the system

$$1 = \tilde w_i a_{ii,i}^1 + \sum_{j\neq i}\tilde w_j(a_{ij,i}^1 + a_{ji,i}^1). \tag{3.32}$$

The solution of this system can be written as

$$\tilde w_i = \frac{\Delta_i}{\Delta}, \quad i = 1,\dots,n \tag{3.33}$$

and all $\tilde w_i > 0$, if $\mathrm{sign}\,\Delta_i = \mathrm{sign}\,\Delta$. Here Δ and Δ_i are the determinants

$$\begin{vmatrix} a_{11,1}^1 & a_{12,1}^1 + a_{21,1}^1 & \cdots & a_{1j,1}^1 + a_{j1,1}^1 & \cdots & a_{1n,1}^1 + a_{n1,1}^1 \\ a_{21,2}^1 + a_{12,2}^1 & a_{22,2}^1 & \cdots & a_{2j,2}^1 + a_{j2,2}^1 & \cdots & a_{2n,2}^1 + a_{n2,2}^1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1,n}^1 + a_{1n,n}^1 & a_{n2,n}^1 + a_{2n,n}^1 & \cdots & a_{nj,n}^1 + a_{jn,n}^1 & \cdots & a_{nn,n}^1 \end{vmatrix}, \tag{3.34}$$

$$\begin{vmatrix} a_{11,1}^1 & a_{12,1}^1 + a_{21,1}^1 & \dots & 1 & \dots & a_{1n,1}^1 + a_{n1,1}^1 \\ a_{21,2}^1 + a_{12,2}^1 & a_{22,2}^1 & \dots & 1 & \dots & a_{2n,2}^1 + a_{n2,2}^1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1,n}^1 + a_{1n,n}^1 & a_{n2,n}^1 + a_{2n,n}^1 & \dots & 1 & \dots & a_{nn,n}^1 \end{vmatrix} \tag{3.35}$$

with the ith column $(1,1,\dots,1)^T$, respectively. If $n = 2$, then

$$\begin{cases} \tilde w_1 = \dfrac{a_{22,2}^1 - a_{12,1}^1 - a_{21,1}^1}{a_{11,1}^1 a_{22,2}^1\bigl(1 - \frac{a_{12,1}^1 + a_{21,1}^1}{a_{22.2}^1}\frac{a_{21,2}^1 + a_{12,2}^1}{a_{11,1}^1}\bigr)}, \\ \tilde w_2 = \dfrac{a_{11,1}^1 - a_{12,2}^1 - a_{21,2}^1}{a_{11,1}^1 a_{22,2}^1\bigl(1 - \frac{a_{12,1}^1 + a_{21,1}^1}{a_{22.2}^1}\frac{a_{21,2}^1 + a_{12,2}^1}{a_{11,1}^1}\bigr)} \end{cases} \tag{3.36}$$

which are positive provided that

$$\mathrm{sign}(a_{22,2}^1 - a_{12,1}^1 - a_{21,1}^1) = \mathrm{sign}(a_{11,1}^1 - a_{12,2}^1 - a_{21,2}^1).$$

If $a_{22,2}^1 = a_{12,1}^1 + a_{21,1}^1$ and $a_{11,1}^1 = a_{12,2}^1 + a_{21,2}^1$, that is if $\beta_{22}^1 = \beta_{12}^1\Omega_{12,1} + \beta_{21}^1\Omega_{21,1}$ and $\beta_{11}^1 = \beta_{21}^1\Omega_{21,2} + \beta_{12}^1\Omega_{12,2}$, then from (3.36) we get only one equation for $\tilde w_1$ and $\tilde w_2$,

$$\tilde w_1\beta_{11}^1 + \tilde w_2\beta_{22}^1 = 1. \tag{3.37}$$

Hence, either $\tilde w_1 < (\beta_{11}^1)^{-1}$ or $\tilde w_2 < (\beta_{22}^1)^{-1}$ is a free positive constant. Obviously, $\beta_{11}^1 + \beta_{22}^1 = \beta_{12}^1 + \beta_{21}^1$.

Lemma 3. *Let hypotheses* (3.29) *hold,* $\beta^2_{ij} = \gamma\beta^1_{ij}$ *with* $\beta^k_{ij} = const > 0$. *Assume that* $b \in C^0(\bar{Q}^3)$ *is positive and bounded and sign of determinants* (3.34) *and* (3.35) *is the same. Then under conditions of Lemma 1 system* (3.30) *has unique positive solution* (3.31) *with* $\tilde{w}_i$ *defined by Eq.* (3.33) *if* $n > 2$, *by Eq.* (3.36) *if* $n = 2$ *and* $\operatorname{sign}\Delta_1 = \operatorname{sign}\Delta_2$, *and by Eq.* (3.37) *if* $n = 2$ *and* $\Delta_1 = \Delta_2 = 0$. *In the last case either* $\tilde{w}_1 < (\beta^1_{11})^{-1}$ *or* $\tilde{w}_2 < (\beta^1_{22})^{-1}$ *is a free positive constant.*

Case 2. Let us set

$$\nu^* = \max_{k,i,j}(\sup_{Q^k} \nu^k_i, \sup_{Q^3} \nu^k_{ij}), \quad \sigma^* = \max_{i,j}\sup_{Q^3}\sigma_{ij}, \quad m^* = \max_{i,j}\sup_{(\tau,\infty)^2} m_{ij},$$

$$b^* = \max_{k,i,j}\sup_{Q^3} b^k_{ij}, \quad \Omega^* = \max_{k,i,j}\sup_{Q^3}\Omega_{ki,j}, \quad b^1_* = \min_{i,j}\inf_{Q^3} b^1_{ij},$$

$$m_* = \min_{i,j}\inf_{(\tau,\infty)^2} m_{ij}, \quad \Omega_* = \min_{k,i,j}\inf_{Q^3}\Omega_{ki,j}.$$

By the transform $w^2_i(0) = \gamma w^1_i(0)$, $w^1_i(0) = \tilde{w}_i/\gamma$ system (3.25) can be written in the form

$$\tilde{w}_i = (\tilde{w}_1)^2 p^1_{11,i} + \tilde{w}_1\tilde{w}_2(p^1_{12,i} + p^1_{21,i}) + (\tilde{w}_2)^2 p^1_{22,i}, \quad i = 1, 2.$$

Then by substitution $\tilde{w}_2 = \phi_2\tilde{w}_1$ we reduce this system to

$$\tilde{w}_1 = (p^1_{11,1} + \phi_2(p^1_{12,1} + p^1_{21,1}) + (\phi_2)^2 p^1_{22,1})^{-1},$$

$$\phi_2 = \frac{p^1_{11,2} + \phi_2(p^1_{12,2} + p^1_{21,2}) + (\phi_2)^2 p^1_{22,2}}{p^1_{11,1} + \phi_2(p^1_{12,1} + p^1_{21,1}) + (\phi_2)^2 p^1_{22,1}}.$$

Since all $p^1_{sk,i}$ are continuous in $(\bar{w}^1, \bar{w}^2, \lambda)$ and, for $\phi_2 > 0$, function $\xi_1(\phi_2) := \phi_2(p^1_{11,1} + \phi_2(p^1_{12,1} + p^1_{21,1}) + (\phi_2)^2 p^1_{22,1})$ grows faster than $\xi_2(\phi_2) := p^1_{11,2} + \phi_2(p^1_{12,2} + p^1_{21,2}) + (\phi_2)^2 p^1_{22,2}$ and $\xi_1(0) < \xi_2(0)$, this cubic equation has only one positive continuous in $(\bar{w}^1, \bar{w}^2, \lambda)$ solution $\phi_2 = \phi(\bar{w}^1, \bar{w}^2, \lambda) \in [p_*/p^*, p^*/p_*]$, where by the definition of $p^1_{ij,k}$ and because of Lemma 1 we have

$$p_*(\lambda) = \frac{2m_* b^1_* \Omega_* \exp\{-2\tau(\lambda + \nu^*)\}}{(2\nu^* + \sigma^* + \lambda)(\nu^* + 2m^* + \lambda)^2} \le \min_{i,j,k} p^1_{ij,k}(\lambda),$$

$$p^*(\lambda) = \frac{2m^* b^{1*} \Omega^* \exp\{-2\tau(\lambda + \nu_*)\}}{(2\nu_* + \lambda)(\nu_* + \lambda)^2} \ge \max_{i,j,k} p^1_{ij,k}(\lambda).$$

Then it follows that

$$\begin{cases} \tilde{w}_1 = \dfrac{1}{q(\bar{w}^1, \bar{w}^2, \lambda)}, \quad \tilde{w}_2 = \dfrac{\phi(\bar{w}^1, \bar{w}^2, \lambda)}{q(\bar{w}^1, \bar{w}^2, \lambda)}, \\ q(\bar{w}^1, \bar{w}^2, \lambda) = p^1_{11,1} + \phi(p^1_{12,1} + p^1_{21,1}) + \phi^2 p^1_{22,1} \end{cases} \tag{3.38}$$

$$w^k_i(0) = h^k_i(\bar{w}^1, \bar{w}^2, \lambda)/q(\bar{w}^1, \bar{w}^2, \lambda), \tag{3.39}$$

where

$$h^1_1 = 1/\gamma, \;\; h^1_2 = \phi(\bar{w}^1, \bar{w}^2, \lambda)/\gamma, \;\; h^2_1 = 1, \;\; h^2_2 = \phi(\bar{w}^1, \bar{w}^2, \lambda).$$

Thus the function

$$\frac{p_*(\bar\lambda)}{p^*(\bar\lambda)} \le \frac{p_*(\lambda)}{p^*(\lambda)} = \frac{m_* b_*^1 \Omega_*}{m^* b^{1*} \Omega^*} \exp\{-2\tau(\nu^* - \nu_*)\}\Big(\frac{\lambda+\nu_*}{\lambda+\nu^*+2m^*}\Big)^2 \frac{\lambda+2\nu_*}{\lambda+\sigma^*+2\nu^*}$$
$$\to \frac{m_* b_*^1 \Omega_*}{m^* b^{1*} \Omega^*} \exp\{-2\tau(\nu^* - \nu_*)\}$$

grows monotonically as $\lambda \to \infty$, where $\bar\lambda = \epsilon - \nu_*$, $\epsilon > 0$. This shows that, for $\lambda \in (\bar\lambda, \infty)$, all $h_i^k(\bar w^1, \bar w^2, \lambda)$ have positive lower and finite upper bounds

$$h_* = \frac{p_*(\bar\lambda)}{p^*(\bar\lambda)} \min(1, 1/\gamma), \quad h^* = \frac{p^*(\bar\lambda)}{p_*(\bar\lambda)} \max(1, \gamma). \tag{3.40}$$

Lemma 4. *Under the conditions of Lemma 1 and Case 2 the system* (3.25) *has only one positive solution defined by Eqs.* (3.38) *and* (3.39)*.*

Now we construct the operator $F^\lambda : D \to D$. From (3.15), (3.18), (3.28), (3.31), and (3.39) we have

$$w_i^k(\tau) = \frac{f_i^k(\tau,\lambda) h_i^k(\bar w^1, \bar w^2, \lambda)}{q(\bar w^1, \bar w^2, \lambda)}.$$

From here and by Eq. (3.18) we conclude that $(\bar w^1, \bar w^2)$ must be a solution of the system

$$w_i^k = \frac{z_i^k f_i^k(\tau,\lambda) h_i^k(\bar w^1, \bar w^2, \lambda)}{q(\bar w^1, \bar w^2, \lambda)}, \quad i = 1, \dots, n, \ k = 1, 2. \tag{3.41}$$

Set

$$M_i^k(\bar w^1, \bar w^2, \lambda) = \frac{z_i^k(\cdot, \bar w^s, \lambda) f_i^k(\tau,\lambda) h_i^k(\bar w^1, \bar w^2, \lambda)}{q(\bar w^1, \bar w^2, \lambda)\tilde\rho(\lambda)},$$
$$\bar M^k(\bar w^1, \bar w^2, \lambda) = (M_1^k(\bar w^1, \bar w^2, \lambda), \dots, M_n^k(\bar w^1, \bar w^2, \lambda))$$

with

$$\tilde\rho(\lambda) = \rho(\bar w^1, \bar w^2, \lambda) = \frac{\sum_{k=1}^{2}\sum_{i=1}^{n} \| z_i^k(\cdot, \bar w^s, \lambda) \| \ f_i^k(\tau,\lambda) h_i^k(\bar w^1, \bar w^2, \lambda)}{q(\bar w^1, \bar w^2, \lambda)} \tag{3.42}$$

and define the operator

$$F^\lambda : (\bar w^1, \bar w^2) \to (\bar M^1(\bar w^1, \bar w^2, \lambda), \bar M^2(\bar w^1, \bar w^2, \lambda)).$$

Here $s = 1, 2$, $s \ne k$ and $\lambda > \bar\lambda$ is a free parameter. Obviously, $F^\lambda : D \to D$. Now we prove that F^λ is continuous in D. By Lemma 1 operator $\tilde F^\lambda : (\bar w^1, \bar w^2) \to (\bar z^1(\cdot, \bar w^2, \lambda), \bar z^2(\cdot, \bar w^1, \lambda))$ for fixed $\lambda > \bar\lambda$ is completely continuous in D and $\tilde F^\lambda(D) \subset \tilde D$. Then, for $\lambda > \bar\lambda$, the map $(\bar w^1, \bar w^2) \to q(\bar w^1, \bar w^2, \lambda)$ is also continuous and $q(\bar w^1, \bar w^2, \lambda) > 0$. Thus, for $\lambda > \bar\lambda$ we have that $\tilde\rho(\lambda) > 0$ and the map $(\bar w^1, \bar w^2) \to \rho(\bar w^1, \bar w^2, \lambda)$ is continuous, as well. All h_i^k are positive and the map $(\bar{w}^1, \bar{w}^2) \to h_i^k(\bar{w}^1, \bar{w}^2, \lambda)$ for fixed λ, k and i is also continuous. Therefore, F^λ is completely continuous and the Schauder fixed point principle yields the existence of at least one fixed point of F^λ in D. This means that the system

$$w_i^k = \frac{z_i^k(\cdot, \bar{w}^s, \lambda) f_i^k(\tau, \lambda) h_i^k(\bar{w}^1, \bar{w}^2, \lambda)}{q(\bar{w}^1, \bar{w}^2, \lambda)\rho(\bar{w}^1, \bar{w}^2, \lambda)}, \; i = 1, \dots, n, \; k, s = 1, 2, \; s \neq k$$

with $\lambda \geq \bar{\lambda}$ has at least one solution $(\bar{w}^1(\cdot, \lambda), \bar{w}^2(\cdot, \lambda)) \in D$. If there exists $\tilde{\lambda} > \bar{\lambda}$ such that $\tilde{\rho}(\tilde{\lambda}) = 1$, then this system coincides with (3.41) for $\lambda = \tilde{\lambda}$ and the function $(\bar{w}^1(\cdot, \tilde{\lambda}), \bar{w}^2(\cdot, \tilde{\lambda}))$ represents a solution of (3.41).

Now we get conditions for the existence or nonexistence of $\tilde{\lambda}$. Set

$$h_* = \min_{k,i} h_i^k > 0, \quad h^* = \max_{k,i} h_i^k \; \text{ in case 1},$$

$$p^1(\lambda) = \sup_{y \in [\tau, \infty)} \int_\tau^\infty z^{1*}(x, \lambda) dx \int_0^\infty \exp\{-\lambda \tau_3\} b(x + \tau_3, y + \tau_3, \tau_3) \times K(x + \tau_3, y + \tau_3, \tau_3, 0)\, d\tau_3,$$

$$p^2(\lambda) = \sup_{x \in [\tau, \infty)} \int_\tau^\infty z^{2*}(y, \lambda) dy \int_0^\infty \exp\{-\lambda \tau_3\} b(x + \tau_3, y + \tau_3, \tau_3) \times K(x + \tau_3, y + \tau_3, \tau_3, 0)\, d\tau_3,$$

$$\tilde{\rho}_*(\lambda) = \sum_{i=1}^n h_i^1 (f^2(\tau, \lambda) p^2(\lambda))^{-1} + \sum_{i=1}^n h_i^2 (f^1(\tau, \lambda) p^1(\lambda))^{-1} \; \text{ in case 1},$$

$$\tilde{\rho}_*(\lambda) = \frac{2h_*}{m^* b^{1*} \Omega^*} \Big(\frac{p_*}{p_* + p^*} \Big)^2 (\lambda + \nu^*)^2 \frac{\lambda + \nu_*}{\lambda + \nu^* + 2m^*} \; \text{ in case 2},$$

$$\tilde{\rho}^*(\lambda) = \sum_{i=1}^n \sum_{k=1}^2 h_i^k f_i^k(\tau, \lambda) \parallel z^{k*} \parallel / q_*(\lambda) \; \text{ in case 1},$$

$$\tilde{\rho}^*(\lambda) = \sum_{i=1}^2 \sum_{k=1}^2 h^* f_i^k(\tau, \lambda) \parallel z^{k*} \parallel / q_*(\lambda) \; \text{ in case 2},$$

$$q_*(\lambda) = f^1(\tau, \lambda) f^2(\tau, \lambda) \int_\tau^\infty z_*^1(x, \lambda) dx \int_\tau^\infty z_*^2(y, \lambda) dy \int_0^\infty \exp\{-\lambda \tau_3\} \times b(x + \tau_3, y + \tau_3, \tau_3) K(x + \tau_3, y + \tau_3, \tau_3, 0)\, d\tau_3 \; \text{ in case 1},$$

$$q_*(\lambda) = p^1_{11,1*} + \frac{p_*}{p^*}(p^1_{12,1*} + p^1_{21,1*}) + (\frac{p_*}{p^*})^2 p^1_{22,1*} \; \text{ in case 2},$$

$$q^*(\lambda) = f^1(\tau, \lambda) f^2(\tau, \lambda) \int_\tau^\infty \int_0^\infty \exp\{-\lambda \tau_3\} z^{1*}(x, \lambda) dx \int_\tau^\infty z^{2*}(y, \lambda) dy \times b(x + \tau_3, y + \tau_3, \tau_3) K(x + \tau_3, y + \tau_3, \tau_3, 0) d\tau_3 \; \text{ in case 1},$$

$$q^*(\lambda) = p^{1*}_{11,1} + \frac{p^*}{p_*}(p^{1*}_{12,1} + p^{1*}_{21,1}) + (\frac{p^*}{p_*})^2 p^{1*}_{22,1} \; \text{ in case 2}.$$

Here

$$p^1_{sj,i*} = f^1_s(\tau,\lambda) f^2_j(\tau,\lambda) \int_\tau^\infty z^1_{s*}(x,\lambda)dx \int_\tau^\infty z^2_{j*}(y,\lambda)\kappa^1_{sj,1}(x,y,\lambda)\,dy,$$

$$p^{1*}_{sj,i} = f^1_s(\tau,\lambda) f^2_j(\tau,\lambda) \int_\tau^\infty z^{1*}_s(x,\lambda)dx \int_\tau^\infty z^{2*}_j(y,\lambda)\kappa^1_{sj,1}(x,y,\lambda)\,dy.$$

In Case 2, constants h_* and h^* are defined by (3.40). Equations (3.27), (3.38), and (3.39) show that $\tilde\rho(\lambda) \ge \tilde\rho_*(\lambda) \to \infty$ as $\lambda \to \infty$. If $\tilde\rho(\bar\lambda) < 1$, then there exists $\tilde\lambda > \bar\lambda$, such that $\tilde\rho(\tilde\lambda) = 1$, since $\tilde\rho \in C^0(\bar\lambda,\infty)$. If $\min\limits_{\lambda\in[\bar\lambda,\lambda_1]} \tilde\rho(\lambda) > 1$, where λ_1 is a unique root of $\tilde\rho_*(\lambda) = 1$, then there is no such $\tilde\lambda > \bar\lambda$ that $\tilde\rho(\tilde\lambda) = 1$

Theorem 1. *Let conditions of one of Lemmas 2, 3, and 4 be satisfied. Then the following assertions are true:*

- *If $\tilde\rho(\bar\lambda) \le 1$, then system* (3.7), (3.9), (3.10) *has at least one positive solution $(\bar w^1(\tau_1,\lambda), \bar w^2(\tau_2,\lambda))$ with $\tilde\lambda \ge \bar\lambda$, such that $w^k_i \in C^0(\bar Q^k)\cap C^1(Q^k)\cap L^1(0,\infty)$,*

  $$z^k_*(\tau_k,\tilde\lambda) f^k(\tau,\tilde\lambda) h_*/q^*(\tilde\lambda) \le w^k_i(\tau_k,\lambda) \le z^{k*}(\tau_k,\tilde\lambda) f^k(\tau,\tilde\lambda) h^*/q_*,$$

  where $k = 1,2$, $i = 1,\dots,n$. If conditions of Lemma 3 are satisfied and $n = 2$, $\Delta_1 = \Delta_2 = 0$, then this solution depends on one positive parameter (see (3.37)*).*

- *If $\min\limits_{\lambda\in[\bar\lambda,\lambda_1]} \tilde\rho(\lambda) > 1$ with a unique root, λ_1 of $\tilde\rho_*(\lambda) = 1$, then system* (3.7), (3.9), (3.10) *has no positive solutions for $\lambda \ge \bar\lambda$.*

Obviously, conditions of this theorem are satisfied if the inequalities $\tilde\rho^*(\bar\lambda) < 1$ and $\tilde\rho_*(\bar\lambda) > 1$ are satisfied, respectively.

Theorem 2. *Let conditions of Theorem 1 be satisfied and let in addition $m_{is} \in C^1([\tau,\infty)^2)$, σ_{is} and $\nu^k_{is} \in C^{1,1,0}(\bar Q^3)$. Then the following assertions are true:*

- *Under condition $\tilde\rho(\bar\lambda) \le 1$ system* (2.1)–(2.7) *has at least one class of positive solutions of type* (3.1) *which depends on parameter f and posses the properties:*

  $$U^k_i \in C^0(\bar Q^k)\cap C^1(Q^k),\ k = 1,2,\ i = 1,\dots,n,\ U^3_{ij} \in C^0(\bar Q^3)\cap C^1(Q^3),$$

  and $\lambda = \tilde\lambda \ge \bar\lambda$. Under the conditions of Lemma 3 and $n = 2$, $\Delta_1 = \Delta_2 =$ 0 this class depends on two parameters (see (3.37)*).*

- *Under condition $\min\limits_{\lambda\in[\bar\lambda,\lambda_1]} \tilde\rho(\lambda) > 1$ system* (2.1)–(2.7) *has no nontrivial separable solutions of type* (3.1) *for $\lambda \ge \bar\lambda$.*

Definition of q and ρ shows that separable solutions exist if pairs produce sufficiently large number of offsprings.

4 The Case of Constant Vital Rates

In this section, we consider the case where vital rates satisfy the following conditions:

(H) ν_i^k and ν_{sj}^k $\forall k, i, s,$ *and* j *are positive constants and do not depend on the sex and confession number*, b_{sj}^k, $\forall k, s, j$ are positive constants and do not depend on k, m_{sj} and σ_{sj}, $\forall s$ *and* j *are positive constants and do not depend on the confession number*, $\Omega_{sj,i}$ *are a positive constants.*

Then w_i^k is also independent of k. Set

$$\sigma = \sigma_{sj}, \quad b_{sj}{=}b_{sj}^k, \quad \nu = \nu_i^k = \nu_{sj}^k, \quad m = m_{sj}, \quad w_i = w_i^k.$$

Equations (3.12), (3.13), and (3.14) can be written in the form

$$\frac{dw_i}{d\tau} = -(\nu + m + \lambda)w_i + m(\nu + \sigma)\exp\{-(2\nu + \sigma + \lambda)\tau_1\} \times \int_\tau^{\tau_1} \exp\{(2\nu + \sigma + \lambda)x\}w_i(x)\,dx, \quad \tau_1 > \tau, \tag{4.1}$$

$$w_i(\tau) = w_i(0)\exp\{-(\nu + \lambda)\tau\}, \tag{4.2}$$

$$w_i(0) = \frac{m}{2\nu + \sigma + \lambda}\sum_{s,j=1}^{n} b_{sj}\Omega_{sj,i}\int_\tau^\infty w_s(x)dx\int_\tau^\infty w_j(y)\,dy,$$

$$\lambda + 2\nu + \sigma > 0. \tag{4.3}$$

Letting $z_i(\tau_1) = w_i(\tau_1)\exp\{(2\nu + \sigma + \lambda)\tau_1\}$, we get the integro-differential equation

$$z_i' - (\nu + \sigma - m)z_i - m(\nu + \sigma)\int_\tau^{\tau_1} z_i(x)\,dx = 0,$$

which by Eq. (4.2) reduces to

$$z_i'' + (m - \nu - \sigma)z_i' - m(\nu + \sigma)z_i = 0, \quad \tau_1 > \tau = 0,$$

$$z_i(\tau) = w_i(0)\exp\{(\nu + \sigma)\tau\}, \; z_i'(\tau) = (\nu + \sigma - m)z_i(\tau).$$

This problem has the solution

$$z_i(\tau_1) = \frac{w_i(0)}{\nu + \sigma + m}\big((\nu + \sigma)\exp\{(\nu + \sigma)\tau_1\} + m\exp\{-m\tau_1 + (\nu + \sigma + m)\tau\}\big).$$

Thus

$$\frac{w_i(\tau_1)}{w_i(0)} = \frac{\exp\{-(\nu + \lambda)\tau_1\}}{\nu + \sigma + m}\big(\nu + \sigma + m\exp\{-(\nu + \sigma + m)(\tau_1 - \tau)\}\big). \tag{4.4}$$

Then

$$\int_\tau^\infty w_i(x)dx = w_i(0)A(\lambda),$$

$$A(\lambda) = \frac{\exp\{-(\nu + \lambda)\tau\}}{\nu + \sigma + m}\Big(\frac{\nu + \sigma}{\nu + \lambda} + \frac{m}{2\nu + \sigma + m + \lambda}\Big), \; \lambda + \nu > 0. \tag{4.5}$$

Now from (4.3) it follows that

$$w_i(0) = \frac{mA(\lambda)^2}{2\nu+\sigma+\lambda}\sum_{s,j=1}^{n} b_{sj}\Omega_{sj,i}w_s(0)w_j(0). \tag{4.6}$$

System (4.6) is of the form (3.26) with $q = \dfrac{mA(\lambda)^2}{2\nu+\sigma+\lambda}$ and $a^k_{sj,i} = b_{sj}\Omega_{sj,i}$ and therefore has at least one positive solution

$$w_i(0) = \frac{h_i(2\nu+\sigma+\lambda)}{mA(\lambda)^2},\ h_i = \frac{\phi_i}{\sum\limits_{s,j=1}^{n}\phi_s\phi_j b_{sj}\Omega_{sj,1}},\ i=1,\dots,n;\ \phi_1 = 1. \tag{4.7}$$

By using $2\sum\limits_{i=1}^{n}\int\limits_{\tau}^{\infty} w_i(x)\,dx = 1$ and (4.5)–(4.7) we get the equation for λ:

$$2\sum_{i=1}^{n} h_i(2\nu+\sigma+\lambda)/m = A(\lambda), \tag{4.8}$$

which has a root

$$-\nu < \tilde\lambda < 0, \quad \text{if} \quad 2\sum_{i=1}^{n} h_i(2\nu+\sigma)/m > A(0) := \frac{(2\nu+\sigma)\exp\{-\nu\tau\}}{\nu(2\nu+\sigma+m)},$$

$$\tilde\lambda = 0, \quad \text{if} \quad 2\sum_{i=1}^{n} h_i(2\nu+\sigma)/m = A(0),$$

$$\tilde\lambda > 0, \quad \text{if} \quad 2\sum_{i=1}^{n} h_i(2\nu+\sigma)/m < A(0).$$

Theorem 3. *Under the hypothesis* (H) *system* (3.2)–(3.7) *has the unique solution of the form* (4.4) *where* $(w_1(0),\dots,w_n(0))$ *is defined by Eqs.* (4.7) *with at least one set* $(h_1,\dots,h_n)$ *of positive constants* $h_i, i=1,\dots,n$*. For each set* $(h_1,\dots,h_n)$*, the unique* $\tilde\lambda$ *is determined by Eq.* (4.8)*.*

5 Proofs of Lemmas 1 and 2

Proof of Lemma 1. We first prove that $G^1_i(\tau_1,x,\lambda) \le 2m^*\|\bar w^2\|$ and is continuous for sufficiently large λ. For $\lambda > -\nu_*$ and $\bar w^2 \in L^1_+(\tau,\infty)$, by definition we get

$$\begin{aligned} G^1_i(\tau_1,x,\lambda) &= \int_x^{\tau_1} R^1_i(\eta,x,\bar w^2,\lambda)\exp\Big\{-\int_\eta^{\tau_1} l^1_i(\xi,\lambda)d\xi\Big\}d\eta \le \int_x^{\tau_1} R^1_i(\eta,x,\bar w^2,\lambda)\,d\eta \\ &= \int_x^{\tau_1} d\eta \int_\tau^\infty \sum_{s=1}^{n} w^2_s(y)K_{is}(x,y,\eta-x,\lambda)(\nu^2_{is}+\sigma_{is})\big|_{(\eta,y+\eta-x,\eta-x)}\,dy \\ &= \int_\tau^\infty \sum_{s=1}^{n} w^2_s(y)dy \int_x^{\tau_1} K_{is}(x,y,\eta-x,\lambda)(\nu^2_{is}+\sigma_{is})\big|_{(\eta,y+\eta-x,\eta-x)}\,d\eta \end{aligned}$$

$$\leq \int_{\tau}^{\infty} \sum_{s=1}^{n} 2m_{is}(x,y) w_s^2(y)\, dy \int_{x}^{\tau_1} \exp\{-\int_{0}^{\eta-x} (\nu_{is}^2 + \sigma_{is})\big|_{(\xi+x,\xi+y,\xi)}\, d\xi\}$$

$$\times\, (\nu_{is}^2 + \sigma_{is})\big|_{(\eta,y+\eta-x,\eta-x)}\, d\eta \leq \sum_{s=1}^{n} \int_{\tau}^{\infty} w_s^2(y) 2m_{is}(x,y)\, dy \leq 2m^* \|\bar{w}^2\|.$$

Hence,

$$G_i^1(\tau_1, x, \lambda) \leq \int_{x}^{\tau_1} g_i^1(\eta, x, \lambda) d\eta \leq 2m^* \|\bar{w}^2\| \tag{5.1}$$

for $\lambda > -\nu_*$ and $\bar{w}^2 \in L_+^1(\tau, \infty)$, and

$$G_i^2(x, \tau_2, \lambda) \leq \int_{x}^{\tau_2} g_i^2(x, \eta, \lambda) d\eta \leq 2m * \|\bar{w}^1\| \tag{5.2}$$

for $\lambda > -\nu_*$ and $\bar{w}^1 \in L_+^1(\tau, \infty)$. It is easy to see that, due to hypotheses of Lemma 1, $G_i^1(\tau_1, x, \lambda)$, $l_i^1(\tau_1, \lambda)$, $G_i^2(x, \tau_2, \lambda)$, $l_i^2(\tau_2, \lambda)$, and $l_i^2(\tau_2, \lambda)$ are continuous functions. Therefore, Volterra equations (3.23) and (3.24) have a unique positive global solution. Obviously, it satisfies Eqs. (3.19) and (3.20) and $z_i^k \in C^0([\tau, \infty)) \cap C^1((\tau, \infty))$ for $\lambda > -\nu_*$.

Now we prove inequalities $z_{i*}^k \leq z_i^k \leq z_i^{k*}$. Let $(\bar{w}^1, \bar{w}^1) \in D$. The lower estimate of z_i^k follows easily from Eqs. (3.25) and (3.26). It remains to prove the upper estimate. Define

$$Z_i^1(\tau_1) = z_i^1(\tau_1) + \int_{\tau}^{\tau_1} z_i^1(x) dx \int_{\tau}^{\infty} \sum_{s=1}^{n} w_s^2(y) K_{is}(x, y, \tau_1 - x, \lambda)\, dy, \quad Z_i^1(\tau) = 1,$$

$$Z_i^2(\tau_2) = z_i^2(\tau_2) + \int_{\tau}^{\tau_2} z_i^2(x) dx \int_{\tau}^{\infty} \sum_{s=1}^{n} w_s^1(y) K_{si}(y, x, \tau_2 - x, \lambda)\, dy, \quad Z_i^2(\tau) = 1.$$

Taking into account Eqs. (3.19) and (3.20) and the definition of K_{is}, r_i^k, l_i^k, R_i^k, and g_i^k we get

$$\frac{dZ_i^1}{d\tau_1} = \frac{dz_i^1}{d\tau_1} + z_i^1 \int_{\tau}^{\infty} \sum_{s=1}^{n} w_s^2(y) K_{is}(\tau_1, y, 0, \lambda)\, dy$$

$$- \int_{\tau}^{\tau_1} z_i^1(x)\, dx \int_{\tau}^{\infty} \sum_{s=1}^{n} w^2(y) K_{is}(x, y, \tau_1 - x, \lambda)(\lambda + \nu_{is}^1 + \nu_{is}^2 + \sigma_{is})\big|_{(\tau_1, \tau_1+y-x, \tau_1-x)} dy$$

$$= -(\lambda + \nu_i^1 + r_i^1) z_i^1 + \int_{\tau}^{\tau_1} z_i^1(x) g_i^1(\tau_1, x, \lambda)\, dx + z_i^1 r_i^1(\tau_1, \bar{w}^2)$$

$$- \int_\tau^{\tau_1} z_i^1(x)dx \int_\tau^\infty \sum_{s=1}^n w^2(y)K_{is}(x,y,\tau_1-x,\lambda)(\lambda+\nu_{is}^1(\tau_1,\tau_1+y-x,\tau_1-x)dy$$

$$- \int_\tau^{\tau_1} dz_i^1(x)R_i^1(\tau_1,x,\bar{w}^2,\lambda)\,dx = -(\lambda+\nu_i^1)z_i^1$$

$$- \int_\tau^{\tau_1} z_i^1(x)dx \int_\tau^\infty \sum_{s=1}^n w^2(y)K_{is}(x,y,\tau_1-x,\lambda)(\lambda+\nu_{is}^1(\tau_1,\tau_1+y-x,\tau_1-x)dy$$

$$\le -(\lambda+\nu_*)Z_i^1,\ Z_i^1(\tau)=1.$$

Hence, $z_i^1 \le Z_i^1 \le \exp\{-(\lambda+\nu_*)(\tau_1-\tau)\}$ and similarly $z_i^2 \le Z_i^2 \le \exp\{-(\lambda+\nu_*)(\tau_2-\tau)\}$.

Now we find the upper estimate for $\|dz_i^k/d\tau_k\|$. Define

$$\Pi_i^k(\tau_k,\lambda) = \exp\Big\{-\int_\tau^{\tau_k} l_i^k(x,\lambda)\,dx\Big\}.$$

Since

$$\frac{d\Pi_i^k(\tau_k,\lambda)}{d\tau_k} = -l_i^k(\tau_k,\lambda)\Pi_i^k(\tau_k,\lambda) < 0 \text{ for } \lambda > -\nu_*,$$

from (3.23) it follows that

$$\Big|\frac{dz_i^1}{d\tau_1}\Big| \le -\frac{d\Pi_i^1}{d\tau_1}\Big\{1+\int_\tau^{\tau_1}(\Pi_i^1(\eta,\lambda))^{-1}d\eta\int_\tau^\eta z_i^1(x)g_i^1(\eta,x,\lambda)dx\Big\}$$
$$+\int_\tau^{\tau_1} z_i^1(x)g_i^1(\tau_1,x,\lambda)\,dx.$$

Then we get

$$\Big\|\frac{dz_i^1}{d\tau_1}\Big\| \le 1 - \int_\tau^\infty \frac{d\Pi_i^1}{d\tau_1}d\tau_1\int_\tau^{\tau_1}(\Pi_i^1(\eta,\lambda))^{-1}d\eta\int_\tau^\eta z_i^1(x)g_i^1(\eta,x,\lambda)\,dx$$
$$+\int_\tau^\infty d\tau_1\int_\tau^{\tau_1} z^1(x)g_i^1(\tau_1,x,\lambda)\,dx = 1+\int_\tau^\infty d\tau_1\int_\tau^{\tau_1} z_i^1(x)g_i^1(\tau_1,x,\lambda)\,dx$$
$$-\int_\tau^\infty(\Pi_i^1(\eta,\lambda))^{-1}\int_\eta^\infty \frac{d\Pi_i^1(\tau_1,\lambda)}{d\tau_1}\,d\tau_1\int_\tau^\eta z_i^1(x)g_i^1(\eta,x,\lambda)\,dx$$
$$= 1+2\int_\tau^\infty d\tau_1\int_\tau^{\tau_1} z_i^1(x)g_i^1(\tau_1,x,\lambda)\,dx$$
$$= 1+2\int_\tau^\infty z_i^1(x)\,dx\int_x^\infty g_i^1(\tau_1,x,\lambda)\,d\tau_1, \quad \lambda > -\nu_*.$$

Similarly,

$$\Big\|\frac{dz_i^2}{d\tau_2}\Big\| \le 1 + 2\int\limits_{\tau}^{\infty} z_i^2(x)\,dx \int\limits_{x}^{\infty} g_i^2(x,\tau_2,\lambda)\,d\tau_2, \quad \lambda > -\nu_*.$$

From here by estimates (5.1) and (5.2) we get

$$\Big\|\frac{dz_i^j}{d\tau_1}\Big\| \le 1 + 4m^*\|z_i^j\| \le 1 + \frac{4m^*}{\lambda+\nu_*}, \quad j=1,2,$$

since $(\bar w^1, \bar w^2) \in D$ and $\lambda > -\nu_*$. Therefore, by the Riesz [6] criterion set $\tilde D$ is compact.

It remains to prove the continuity of the operator $\tilde F^\lambda$ in D. Let $(\bar w^{11}, \bar w^{21})$ and $(\bar w^{12}, \bar w^{22}) \in D$. Let corresponding pairs be $(\bar z^1(\cdot,\bar w^{21}), \bar z^2(\cdot,\bar w^{11}))$ and $(\bar z^1(\cdot,\bar w^{22}), \bar z^2(\cdot,\bar w^{12}))$. Set

$$\Delta\bar w^k = \bar w^{k2} - \bar w^{k1}, \;\; \Delta z_i^k = z_i^k(\cdot,\bar w^{22},\cdot) - z_i^k(\cdot,\bar w^{21},\cdot), \;\; k=1,2.$$

From Eqs. (3.21) and (3.22) it follows that

$$\frac{d\Delta z_i^1}{d\tau_1} = -(\lambda+\nu_i^1(\tau_1)+r_i^1(\tau_1,\bar w^{22}))\Delta z_i^1 - z_i^{11} r_i^1(\tau_1,\Delta\bar w^2)$$
$$+\int\limits_{\tau}^{\tau_1} \Delta z_i^1(x) R_i^1(\tau_1,x,\bar w^{22},\lambda)dx + \int\limits_{\tau}^{\tau_1} z_i^{11}(x) R_i^1(\tau_1,x,\Delta\bar w^2,\lambda)dx, \; \Delta z_i^1(\tau)=0,$$
$$\frac{d\Delta z_i^2}{d\tau_2} = -(\lambda+\nu_i^2(\tau_2)+r_i^2(\tau_2,\bar w^{12}))\Delta z_i^2 - z_i^{21} r_i^2(\tau_2,\Delta\bar w^1)$$
$$+\int\limits_{\tau}^{\tau_2} \Delta z_i^2(x) R_i^2(x,\tau_2,\bar w^{12},\lambda)dx + \int\limits_{\tau}^{\tau_2} z_i^{21}(x) R_i^2(x,\tau_2,\Delta\bar w^1,\lambda)dx, \; \Delta z_i^2(\tau)=0.$$

Hence

$$\Delta z_i^1 = \int\limits_{\tau}^{\tau_1} \exp\{-\int\limits_{\eta}^{\tau_1}(\lambda+\nu_i^1(y)+r_i^1(y,\bar w^{22}))dy\}\{\int\limits_{\tau}^{\eta}[\Delta z_i^1(x) R_i^1(\eta,x,\bar w^{22},\lambda)$$
$$+ z_i^{11}(x) R_i^1(\eta,x,\Delta\bar w^2,\lambda)]\,dx - z_i^{11}(\eta) r_i^1(\eta,\Delta\bar w^2)\}\,d\eta$$

or

$$\Delta z_i^1 = \int\limits_{\tau}^{\tau_1} dx \int\limits_{x}^{\tau_1} \exp\{-\int\limits_{\eta}^{\tau_1}(\lambda+\nu_i^1(y)+r_i^1(y,\bar w^{22}))dy\}$$
$$\times [\Delta z_i^1(x) R_i^1(\eta,x,\bar w^{22},\lambda) + z_i^{11}(x) R_i^1(\eta,x,\Delta\bar w^2,\lambda)]\,d\eta$$
$$-\int\limits_{\tau}^{\tau_1} \exp\{-\int\limits_{\eta}^{\tau_1}(\lambda+\nu_i^1(y)+r_i^1(y,\bar w^{22}))\} z_i^{11}(\eta) r_i^1(\eta,\Delta\bar w^2)\,d\eta.$$

From here by using Eqs. (5.1) and (3.15), for $\lambda > -\nu_*$, we get the inequality

$$|\Delta z_i^1| \le 2m^* \int_\tau^{\tau_1} |\Delta z_i^1(x)|dx + 2m^* \|\Delta \bar{w}^2\| \frac{1+e^{-1}}{\lambda+\nu_*},$$

which shows that

$$|\Delta z_i^1| \le \frac{2m^*(1+e^{-1})}{\lambda+\nu_*} \|\bar{w}^2\| \exp\{2m^*(\tau_1-\tau)\}$$

and similarly,

$$|\Delta z_i^2| \le \frac{2m^*(1+e^{-1}}{\lambda+\nu_*} \|\bar{w}^1\| \exp\{2m^*(\tau_2-\tau)\}.$$

But $|\Delta z_i^1| \le 2z_i^{1*} = 2\exp\{-(\lambda+\nu_*)(\tau_1-\tau)\}$ for $\lambda > -\nu_*$. Hence,

$$\|\Delta z_i^1\| \le \int_\tau^\infty |\Delta z_i^1| d\tau_1 \le \frac{2m^*(1+e^{-1})}{\lambda+\nu_*} \|\bar{w}^2\| \int_\tau^{\tilde{\tau}_*(\lambda)} \exp\{2m^*(\tau_1-\tau)\}d\tau_1$$

$$+ 2\int_{\tilde{\tau}_*(\lambda)}^\infty \exp\{2m^*(\tau_1-\tau)\}d\tau_1 = c_1(\lambda) \parallel \Delta \bar{w}^2 \parallel + c_2(\lambda),$$

where

$$c_1(\lambda) = \frac{1+e^{-1}}{\lambda+\nu_*}(\exp\{2m^*(\tilde{\tau}_*(\lambda)-1)\}-1) \le c_1(\bar{\lambda}),$$

$$c_2(\lambda) = \frac{2}{\lambda+\nu_*}\exp\{-(\lambda+\nu_*)(\tilde{\tau}_*(\lambda)-\tau)\} \le c_2(\bar{\lambda}),\ \lambda > \bar{\lambda}.$$

Choose $\tilde{\tau}_*(\lambda)$ and $\|\Delta \bar{w}^2\|$ such that $c_2(\lambda) < \epsilon_1/2$ for $\lambda \ge \bar{\lambda}$ and $\|\Delta \bar{w}^2\| < \epsilon_1/2c_1(\bar{\lambda})$, $\epsilon_1 > 0$. Then $\|\Delta z_i^2\| \le \epsilon_1$ for $\lambda \ge \bar{\lambda}$. Similarly, $\|\Delta z_i^2\| \le \epsilon_1$ for $\lambda \ge \bar{\lambda}$. Therefore, operator $\tilde{F}^\lambda$ is continuous in D for $\lambda \ge \bar{\lambda}$. Thus, we conclude that operator $\tilde{F}^\lambda$ is completely continuous. The proof of Lemma 1 is completed.

Proof of Lemma 2. Letting $w_i^k(0) = w_1^1(0)\phi_i^k$, from Eq. (3.26) we get

$$w_1^1(0) = \Big\{ \sum_{i,j}^n \phi_s^1 \phi_j^2 a_{sj,1}^1 q(\bar{w}^1, \bar{w}^2, \lambda) \Big\}^{-1}, \tag{5.3}$$

$$\phi_1^1 = 1, \quad \phi_i^k = \frac{\sum\limits_{s,j=1}^n \phi_s^1 \phi_j^2 a_{sj,i}^k}{\sum\limits_{s,j=1}^n \phi_s^1 \phi_j^2 a_{sj,1}^1}, \quad i = 2, \dots, n, \ \ k = 1, 2. \tag{5.4}$$

We look for positive $\phi_s^1, \phi_j^2\, \forall s, j$. We consider the case $a_{sj,i}^k > 0$, i.e. $\Omega_{sj,i} > 0$ $\forall s, j, k$. It is evident that $\phi_i^k \in (\phi_*^k, \phi^{k*})$, $i = 2, \dots, n$, where

$$\phi_*^k = \min_{s,j,i} a_{sj,i}^k / \max_{sj,i} a_{sj,1}^k, \quad \phi^{k*} = \max_{s,j,i} a_{sj,i}^k / \min_{sj,i} a_{sj,1}^k.$$

From (5.4) with $k=1$ and $i=2$ we get the quadratic equation for ϕ_2^1,

$$\phi_2^1(\phi_2^1\sum_{j=1}^n\phi_j^2a_{2j,1}^1+\sum_{s\neq 2}\sum_{j=1}^n\phi_s^1\phi_j^2a_{sj,1}^1)=\phi_2^1\sum_{j=1}^n\phi_j^2a_{2j,2}^1+\sum_{s\neq 2}\sum_{j=1}^n\phi_s^1\phi_j^2a_{sj,2}^1,$$

which has a unique positive continuous solution $\phi_2^1=\Phi_2^1(\phi_3^1,\dots,\phi_n^2)$. Letting

$$y_3^1(\phi_3^1;\Phi_2^1,\phi_4^1,\dots,\phi_n^2)=(\phi_3^1)^2\sum_{j=1}^n\phi_j^2a_{3j,1}^1+\phi_3^1\sum_{s\neq 3}\sum_{j=1}^n\phi_s^1\phi_j^2a_{sj,1}^1\big|_{\phi_2^1=\Phi_2^1(\phi_3^1,\dots,\phi_n^2)},$$

$$z_3^1(\phi_3^1;\Phi_2^1,\phi_4^1,\dots,\phi_n^2)=\phi_3^1\sum_{j=1}^n\phi_j^2a_{3j,3}^1+\sum_{s\neq 3}\sum_{j=1}^n\phi_s^1\phi_j^2a_{sj,3}^1\big|_{\phi_2^1=\Phi_2^1(\phi_3^1,\dots,\phi_n^2)}$$

from (5.4) we derive the equation for ϕ_3^1:

$$y_3^1(\phi_3^1;\Phi_2^1(\phi_3^1,\phi_4^1,\dots,\phi_n^2),\phi_4^1,\dots,\phi_n^2)=z_3^1(\phi_3^1;\Phi_2^1(\phi_3^1,\dots,\phi_n^2),\phi_4^1,\dots,\phi_n^2). \tag{5.5}$$

Since

$$y_3^1(\phi_3^1;\phi_*^1,\phi_4^1,\dots,\phi_n^2)\le y_3^1(\phi_3^1;\Phi_2^1(\phi_3^1,\dots,\phi_n^2),\phi_4^1,\dots,\phi_n^2)\le y_3^1(\phi_3^1;\phi^{1*},\phi_4^1,\dots,\phi_n^2),$$
$$z_3^1(\phi_3^1;\phi_*^1,\phi_4^1,\dots,\phi_n^2)\le z_3^1(\phi_3^1;\Phi_2^1(\phi_3^1,\dots,\phi_n^2),\phi_4^1,\dots,\phi_n^2)\le z_3^1(\phi_3^1;\phi^{1*},\phi_4^1,\dots,\phi_n^2),$$

and $\Phi_2^1(\phi_3^1,\dots,\phi_n^2)$ is continuous, the graph analysis shows that Eq. (5.5) has at least one positive continuous solution $\phi_3^1=\Phi_3^1(\phi_4^1,\dots,\phi_n^2)$. Therefore,

$$\phi_2^1=\Phi_2^1\big(\Phi_3^1(\phi_4^1,\dots,\phi_n^2),\phi_4^1,\dots,\phi_n^2\big).$$

We proceed this argument getting from (5.4) with $i=n$ the equation for ϕ_n^1,

$$\phi_n^1\Big(\phi_n^1\sum_{j=1}^n\phi_j^2a_{nj,1}^1+\sum_{s=1}^{n-1}\sum_{j=1}^n\phi_s^1\phi_j^2a_{sj,1}^1\Big)=\phi_n^1\sum_{j=1}^n\phi_j^2a_{nj,n}^1+\sum_{s=1}^{n-1}\sum_{j=1}^n\phi_s^1\phi_j^2a_{sj,n}^1,$$

where $\phi_2^1,\phi_3^1,\dots,\phi_{n-1}^1$ depend only on $\phi_n^1,\phi_1^2,\dots,\phi_n^2$ and are known continuous functions. By the argument applied to Eq.(5.5) we prove that this equation has at least one positive continuous solution $\phi_n^1=\Phi_n^1(\phi_1^2,\dots,\phi_n^2)$. Knowing ϕ_n^1 we determine $\phi_2^1,\dots,\phi_{n-1}^1$ as functions of $\phi_1^2,\dots,\phi_n^2$.

Now we consider (5.4) for $k=2$ and $i=1$ getting the equation for ϕ_1^2:

$$\phi_1^2\Big(\phi_1^2\sum_{s=1}^n\phi_s^1a_{s1,1}^1+\sum_{j=2}^n\sum_{s=1}^n\phi_s^1\phi_j^2a_{sj,1}^1\Big)=\phi_1^2\sum_{s=1}^n\phi_s^1a_{s1,1}^2+\sum_{s=1}^n\sum_{j=2}^n\phi_s^1\phi_j^2a_{sj,1}^2, \tag{5.6}$$

where $\phi_2^1,\dots,\phi_n^1$ are known functions of $\phi_1^2,\dots,\phi_n^2$. By the argument similar to that applied to (5.5) we prove that Eq. (5.6) has at least one positive continuous solution $\phi_1^2=\Phi_1^2(\phi_2^2,\dots,\phi_n^2)$. This allows us to determine $\phi_2^1,\dots.\phi_n^1$ as functions of $\phi_2^2,\dots,\phi_n^2$.

We proceed this argument getting, for $i=n$, the equation

$$\phi_n^2\Big(\phi_n^2\sum_{s=1}^n\phi_s^1a_{sn,1}^1+\sum_{s=1}^n\sum_{j=1}^{n-1}\phi_s^1\phi_j^2a_{sj,1}^1\Big)=\phi_n^2\sum_{s=1}^n\phi_s^1a_{sn,n}^2+\sum_{j=1}^{n-1}\sum_{s=1}^n\phi_s^1\phi_j^2a_{sj,n}^2, \tag{5.7}$$

where $\phi_2^1, \dots, \phi_n^1, \phi_1^2$, ϕ_{n-1}^2 are known continuous functions of ϕ_n^2. By using the argument applied to Eq.(5.4) we prove that Eq.(5.7) has at least one positive continuous solution ϕ_n^2. This allows us to determine $\phi_2^1, \dots, \phi_{n-1}^2$.

Thus, when $a_{sj,i}^k > 0$, $\forall s, j, k$, system (3.26) has a positive solution

$$(w_1^1(0), w^1(0)\phi_2^1 \dots, w_1^1(0)\phi_n^1, w_1^1(0)\phi_1^2, \dots, w_1^1(0)\phi_n^2)$$

with at least one positive set $(\phi_2^1, \dots \phi_n^1, \phi_1^2, \dots \phi_n^2)$ independent of λ and $w_1^1(0)$ defined by Eq. (5.3). The proof of Lemma 2 is completed.

6 Concluding Remarks

The existence and non-existence of separable solutions are studied to age-sex – religion structured human communities model which forbid any confession change for individuals but let parents choose a religion not necessary their own for their offsprings. The proof of the existence of separable solutions is based on the Schauder fixed-point principle which does not let to prove the uniqueness.

The main problem in study of separable solutions to this model is to prove the existence theorem to system (3.25) and examine the asymptotic behaviour of its solution as λ tends to ∞. Therefore we have studied only some particular cases of this system. Examination of the general case is an open problem.

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