Mathematical Modelling and Analysis ISSN 1392-6292 print
Volume 14 Number 1, 2009, pages 33–42 ISSN 1648-3510 online
Doi:10.3846/1392-6292.2009.14.33-42 www.vgtu.lt/mma/

Nonlinear Spectra: the Neumann Problem

A. Gritsans and F. Sadyrbaev

Daugavpils University
Parades 1, LV-5400, Daugavpils, Latvia
E-mail: `arminge@inbox.lv`
E-mail(*corresp.*): `felix@latnet.lv`

Received October 1, 2008; revised January 10, 2009; published online February 25, 2009

Abstract. Eigenvalue problems of the form $x'' = -\lambda f(x^+) + \mu g(x^-)$, $x'(a) = 0$, $x'(b) = 0$ are considered. We are looking for (λ, μ) such that the problem (i), (ii) has a nontrivial solution. This problem generalizes the famous Fučík problem for piecewise linear equations. In our considerations functions f and g may be nonlinear. Consequently spectra may differ essentially from those for the Fučík equation.

Key words: nonlinear spectra, jumping nonlinearity, asymptotically asymmetric nonlinearities, Fučík spectrum, Neumann boundary conditions.

1 Formulation of the Problem

In this article we consider equations

$$x'' = -\lambda f(x) \tag{1.1}$$

and

$$x'' = -\lambda f(x^+) + \mu g(x^-), \tag{1.2}$$

where λ and μ are the parameters, $f, g : [0, +\infty) \to [0, +\infty)$ are locally Lipschitz continuous functions such that $f(0) = g(0) = 0$, $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. The boundary conditions are of the Neumann type

$$x'(a) = 0, \quad x'(b) = 0. \tag{1.3}$$

We are looking for such values of λ (resp.: λ and μ) that the problem (1.1), (1.3) (resp.: (1.2), (1.3) has a nontrivial solution.

The spectrum of the first problem (if any) is relatively simple one-dimensional set of points. A report on investigation of this problem and a plenty of interesting results may be found in [4].

The spectrum of the second problem may consist of a set of planar curves as in the case of the Fučík equation

$$x'' = -\lambda x^+ + \mu x^-, \tag{1.4}$$

which is obtained from (1.2) if $f = g = x$. Intensive literature is devoted to spectral problems for the Fučík equation and generalizations. We mention the recent articles [2, 7] which also contains related references. The first branches of the spectrum for the problem (1.4), (1.3) are depicted in Fig. 1.

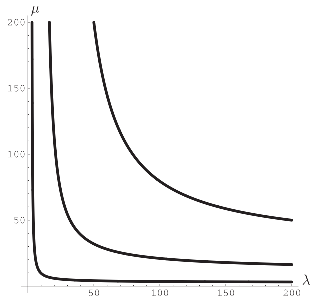

Figure 1. The Neumann problem: classical spectrum.

The spectrum of the problem (1.4), (1.3) for $a = 0, b = 1$ is given by the relations ($i = 1, 2, \ldots$):

$$F_i^{\pm} = \left\{(\lambda; \mu) : \quad \frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{\mu}} = \frac{2(b-a)}{i\pi}\right\}, \quad i = 1, 2, \ldots.$$

The branch F_i^+ (resp: F_i^-) consists of points (λ, μ) such that equation (1.4) has a solution which satisfies the Neumann boundary conditions, has exactly i zeros in (a, b) and the derivative x' at $t = a$ is positive (resp: negative).

The rest of the paper is organized as follows. In Section 2 we consider the one parameter problem. In Section 3 we study problem (1.2), (1.3) and provide the theorem which gives description of the spectrum. The subsection 3.1 is devoted to relatively simple case when the spectrum is similar to the classical one. The subsection 3.2 contains material showing that the spectrum of the problem (1.2), (1.3) may differ essentially from the classical one. In Section 4 an example is considered which shows that branches of the spectrum may be of usual nature (planar hyperbola looking curve) or may contain several disjoint components and some of these components may be even bounded.

2 One-Parameter Problems

Consider the problem

$$x'' + f(x) = 0, \quad x(0) = 0, \quad x'(0) = \alpha > 0.$$

Let $t_1(\alpha)$ be the time map (the first zero function). By rescaling the time t, one easily can show that function $U(\alpha, \lambda) = \frac{1}{\sqrt{\lambda}} t_1\left(\alpha/\sqrt{\lambda}\right)$ defines the time map for the problem

$$x'' + \lambda f(x) = 0, \quad x(0) = 0, \quad x'(0) = \alpha > 0.$$

A respective solution satisfies the following zero Dirichlet boundary conditions

$$x(a) = 0, \quad x(a + U(\alpha, \lambda)) = 0 \tag{2.1}$$

and is symmetric with respect to the middle point of the interval $(a, a+U(\alpha, \lambda))$.

If we consider equation

$$x'' + \lambda f(x^+) - \lambda f(x^-) = 0, \tag{2.2}$$

then also negative valued solutions are allowed and functions $\pm x(t + \frac{b-a}{2}; \alpha, \lambda)$ solve the Neumann problem (2.2), (1.3), where $x(t; \alpha, \lambda)$ is a positive solution of the problem

$$x'' + \lambda f(x) = 0, \quad x(a) = 0, \quad x(b) = 0.$$

If there exist solutions of problem (2.1) with multiple zeros then, due to autonomity of the equation, solutions of the Neumann problem can be constructed in the same interval. More precisely, the following statement is true.

Proposition 1. *Let (α, λ), $\alpha > 0$, $\lambda > 0$ be solutions of the equation*

$$U(\alpha, \lambda) = \frac{1}{n}(b - a), \quad n = 1, 2, \ldots.$$

Then the Dirichlet problem

$$\begin{aligned}&x'' + \lambda f(x) = 0,\\ &x(a) = 0,\ \ x(b) = 0,\ \ x'(a) = \alpha,\ \ x(t) \textit{ has exactly } (n-1) \textit{ zeros in } (a, b)\end{aligned}$$

and the Neumann problem

$$\begin{aligned}&x'' + \lambda f(x^+) - \lambda f(x^-) = 0,\\ &x'(a) = x'(b) = 0,\ \ |x'(z)| = \alpha,\ \ \textit{if } x(z) = 0,\\ &x(t) \textit{ has exactly } n \textit{ zeros in } (a, b)\end{aligned}$$

have solutions.

For instance, equation $U(\alpha, \lambda) = b - a$ defines all pairs (α, λ), for which a positive solution of the Dirichlet problem exists in the interval (a, b) and a solution with exactly one zero in (a, b) exists for the Neumann problem. The values of derivative x' at the zero points of x are equal in modulus to α. For example, for equation

$$x'' = -\lambda x^3,$$

in the interval $(0, 1)$ the respective relation between α and λ, which connects positive solutions of the Dirichlet problem and solutions of the Neumann problem with exactly one zero, is given by

$$\alpha^{1/2}\lambda^{1/4} = 2^{5/4}\int_0^1 \frac{dx}{\sqrt{1 - x^4}}\,.$$

Notice that for any $\alpha > 0$ there exists the respective λ and vice versa.

If we look for the spectrum of the Neumann problem (2.2), (1.3) then we should consider the Dirichlet problem

$$x'' = -\lambda f(x), \quad x(0) = 0, \;\; x(1) = 0.$$

This problem may have a continuous spectrum as the following example of

$$x'' = -\lambda x^3, \quad x(0) = 0, \;\; x(1) = 0$$

shows. Therefore one formulate some normalization condition which prevents continuous spectrum. This additional condition might be $|x'(0)| = 1$.

3 Two-Parameter Problems

Consider the problem

$$x'' = \begin{cases} -\lambda f(x), & \text{if} \quad x \geq 0 \\ \mu g(-x), & \text{if} \quad x < 0, \end{cases} \qquad x'(a) = x'(b) = 0.$$

Suppose that f and g satisfy the conditions (A1) and (A2) respectively.

(A1) A first zero $t_1(\gamma)$ of a solution to the Cauchy problem

$$u'' = -f(u), \quad u(a) = 0, \;\; u'(a) = \gamma$$

exists for any $\gamma > 0$.

(A2) A first zero $\tau_1(\delta)$ of a solution to the Cauchy problem

$$v'' = g(v), \quad v(a) = 0, \;\; v'(a) = -\delta$$

exists for any $\delta > 0$.

Let us introduce the normalization condition $|x'(z)| = 1$, where z is a zero point.

Remark 1. Without the normalization condition the problem may have (and generally has) a continuous spectrum.

We consider the following problem

$$x'' = \begin{cases} -\lambda f(x), & \text{if} \quad x \geq 0 \\ \mu g(-x), & \text{if} \quad x < 0, \\ x'(a) = x'(b) = 0, & |x'(z)| = 1, \; \{z \in (a,b) : x(z) = 0\}. \end{cases} \tag{3.1}$$

Theorem 1. *Let the conditions* **(A1)** *and* **(A2)** *hold with the functions* $t_1(\gamma)$ *and* $\tau_1(\delta)$*. The Fučík spectrum for the problem* (3.1) *is given by the relations* $(i = 1, 2, \ldots)$:

$$F_i^{\pm} = \left\{ (\lambda; \mu) : \; \frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + \frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = \frac{2(b-a)}{i} \right\}. \tag{3.2}$$

Proof. Consider the problem

$$x'' = -\lambda f(x), \quad x'(a) = 1,\ x(a) = 0,$$

where $\lambda > 0$ is fixed. Let $a + 2T_1(\lambda)$ be the first zero of $x(t)$ in $(a, +\infty)$. Then it follows from [1] that $2T_1 = t_1\big(\frac{1}{\sqrt{\lambda}}\big)/\sqrt{\lambda}$. Then function $u(t) = x(t - T_1)$ solves the problem

$$u'' = -\lambda f(u), \quad u'(a) = 0,\ u(a + T_1) = 0,\ u(t) > 0,\ \ t \in (a, a + T_1).$$

Notice that $u'(a + T_1) = -1$. Consider the problem

$$y'' = \mu g(-y), \quad y'(a) = -1,\ y(a) = 0,$$

where $\mu > 0$ is fixed. Then the first zero of $y(t)$ in $(a, +\infty)$ is at $t = a + 2T_2$, where $2T_2 = \tau_1\big(\frac{1}{\sqrt{\mu}}\big)/\sqrt{\mu}$ (see, [1]). Then function $v(t) = y(t - T_1)$ satisfies the conditions

$$v(a + T_1) = 0, \quad v'(a + T_1) = -1$$

and it is a smooth continuation of $u(t)$. The function

$$w(t) = \begin{cases} u(t), & \text{if} \quad t \in [a, a + T_1] \\ v(t), & \text{if} \quad t \in [a + T_1, a + T_1 + T_2], \end{cases}$$

is a C^2-solution of the equation (3.1) with the conditions

$$w'(a) = 0, \quad w'(a + T_1 + T_2) = 0.$$

Besides, this function has exactly one zero (at $t = a + T_1$) in the interval $(a, a+T_1+T_2)$. If λ and μ are such that $a+T_1+T_2 = b$ then $(\lambda, \mu) \in F_1^+$. Thus the relation which defines the first positive branch of the spectrum is given by

$$F_1^+ = \left\{(\lambda; \mu) : \ \frac{1}{2}\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + \frac{1}{2}\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = b - a\right\}.$$

In a similar manner other relations (3.2) can be obtained. $\square$

3.1 Some properties of spectra

Let us consider the functions

$$U(\lambda) := \frac{1}{\sqrt{\lambda}} t_1\Big(\frac{1}{\sqrt{\lambda}}\Big), \quad V(\mu) := \frac{1}{\sqrt{\mu}} \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big),$$

where t_1 and τ_1 are the time maps associated with f and g respectively. Due to Theorem 1 the spectrum of the Neumann problem is a union of pairs (λ, μ) such that

$$\frac{i}{2}U(\lambda) + \frac{i}{2}V(\mu) = b - a, \quad i = 1, 2, \ldots.$$

Proposition 2. *If functions U and V both are monotonically decreasing from $+\infty$ to 0, then the spectrum of the Neumann problem is essentially the classical one.*

Points λ_i (resp: μ_i) of intersection of the graph of $U(\lambda)$ (resp: $V(\mu_i)$) with the straight line $U = b - a$ (resp: $V = b - a$) gives vertical (resp: horizontal) asymptotes on the (λ, μ) plane for the branches $F_i^{\pm}$ of the spectrum, $i = 1, 2, \ldots$. Acting like in [3, 6], one can obtain formulas:

$$U(\lambda) = 2\int_0^{x_+} \frac{dx}{\sqrt{1 - 2\lambda F(x)}},$$

$$U'(\lambda) = -\frac{2}{\lambda}\int_0^{x_+} \left(1 - \frac{F(x)F''(x)}{f^2(x)}\right) \frac{dx}{\sqrt{1 - 2\lambda F(x)}},$$

$$U(\lambda) = 4\int_0^{x_+} \left(\frac{3}{2} - \frac{F(x)F''(x)}{f^2(x)}\right) \sqrt{1 - 2\lambda F(x)}\, dx$$

$$U'(\lambda) = -\frac{2}{\lambda}\int_0^{x_+} \left(3 - 7\frac{F(x)F''(x)}{f^2(x)} + 6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2 - 2\frac{F^2(x)F'''(x)}{f^3(x)}\right) \sqrt{1 - 2\lambda F(x)}\, dx,$$

where x_+ is a maximal value of a solution to the Cauchy problem

$$x'' + f(x) = 0, \quad x(a) = 0, \;\; x'(a) = 1 \text{ and } F(x) = \int_0^x f(s)\, ds.$$

Proposition 3. *If* $1 - F(x)F''(x)/f^2(x) > 0$ *or*

$$3 - 7\frac{F(x)F''(x)}{f^2(x)} + 6\left(\frac{F(x)F''(x)}{f^2(x)}\right)^2 - 2\frac{F^2(x)F'''(x)}{f^3(x)} > 0,$$

then function U is monotonically decreasing.

The dual statement is valid for V where F should be substituted by G, the anti-derivative of g.

3.2 Location of branches

If functions f and g are majorized and minorized by linear functions then the differential inequalities in [5, Ch. 15, Theorem 15.3] lead to the following result. Suppose that

$$k_1^2 x < f(x) < k_2^2 x, \quad m_1^2 x < g(x) < m_2^2 x, \quad x > 0.$$

Let ξ_i be the first zero of a solution to the problem

$$x'' = -k_i^2 x, \quad x(0) = 0, \;\; x'(0) = 1, \quad i = 1, 2 \tag{3.3}$$

and let η_i be the first zero of a solution to the problem

$$x'' = -m_i^2 x, \quad x(0) = 0, \;\; x'(0) = 1, \quad i = 1, 2. \tag{3.4}$$

Proposition 4. $\frac{\xi_2}{\sqrt{\lambda}} < \frac{1}{\sqrt{\lambda}} t_1\left(\frac{1}{\sqrt{\lambda}}\right) < \frac{\xi_1}{\sqrt{\lambda}}$ *for any* $\lambda > 0$.

Proof. The proof follows from comparison of the angular functions for equations $x'' + f(x) = 0$ and (3.3), which results in the inequalities $\xi_2 < t_1 < \xi_1$ for the first zeros. □

The dual proposition states that

Proposition 5. $\frac{\eta_2}{\sqrt{\mu}} < \frac{1}{\sqrt{\mu}} \tau_1\left(\frac{1}{\sqrt{\mu}}\right) < \frac{\eta_1}{\sqrt{\mu}}$ *for any* $\mu > 0$.

Corollary 1. Branch $F_N^{\pm}$ of the spectrum is located in the region between the curves $\Gamma_1 = \{(\lambda, \mu) :\ \lambda > 0, \mu > 0, \frac{\xi_1}{\sqrt{\lambda}} + \frac{\eta_1}{\sqrt{\mu}} = \frac{2(b-a)}{N}\}$ and $\Gamma_2 = \{(\lambda, \mu) :\ \lambda > 0, \mu > 0, \frac{\xi_2}{\sqrt{\lambda}} + \frac{\eta_2}{\sqrt{\mu}} = \frac{2(b-a)}{N}\}$.

The proof follows from representation of the branches in Theorem 1 and propositions given above.

Corollary 2. There are no branches $F_N^{\pm}, F_{N+1}^{\pm}, \ldots$ of the spectrum in the region defined by

$$\frac{\xi_2}{\sqrt{\lambda}} + \frac{\eta_2}{\sqrt{\mu}} > \frac{2(b-a)}{N}.$$

Corollary 3. There are no branches $F_1^{\pm}, \ldots, F_N^{\pm}$ of the spectrum in the region defined by

$$\frac{\xi_1}{\sqrt{\lambda}} + \frac{\eta_1}{\sqrt{\mu}} < \frac{2(b-a)}{N}.$$

Proposition 6. *Let* Υ *be a Jordan curve in the region* $\{(\lambda, \mu) :\ \lambda > 0,\ \mu > 0\}$ *such that*

$$\frac{1}{\sqrt{\lambda}} t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + \frac{1}{\sqrt{\mu}} \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) < \frac{2(b-a)}{N} \quad (\textit{resp.}\ \ >) \tag{3.5}$$

for any $(\lambda, \mu) \in \Upsilon$ *and, at the same time,*

$$\frac{1}{\sqrt{\lambda_*}} t_1\Big(\frac{1}{\sqrt{\lambda_*}}\Big) + \frac{1}{\sqrt{\mu_*}} \tau_1\Big(\frac{1}{\sqrt{\mu_*}}\Big) > \frac{2(b-a)}{N} \quad (\textit{resp.}\ \ <) \tag{3.6}$$

for some $(\lambda_*, \mu_*) \in$ interior Υ. *Then the branch* $F_N^{\pm}$ *has an isolated component in interior* Υ.

Proof. Indeed, the branch $F_N^{\pm}$ is defined by the relation (3.5) where the equality sign "=" replaces ">" or "<"(cf. (3.2)). Therefore there are no points of $F_N^{\pm}$ on Υ. On the other hand, in view of (3.6), there are points of $F_N^{\pm}$ in interior Υ. □

4 Example

Let us consider function $f(x) = 0.25xe^{1.115x^3-1.9x^2+1}$. Function f is concave-convex, strictly increasing and smooth (see Fig. 2). Let us define f for negative x as $f(x) = -f(-x)$. The first zero function (time map) is depicted in Fig. 3 and the following limits are valid:

$$\lim_{\alpha\to 0+} t_1(\alpha) = \frac{2\pi}{\sqrt{e}} = 3.81\ldots, \qquad \lim_{\alpha\to +\infty} t_1(\alpha) = 0.$$

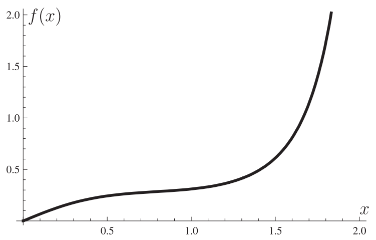

Figure 2. The graph of $f(x)$.

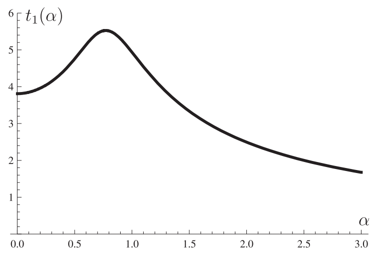

Figure 3. The graph of time map function t_1.

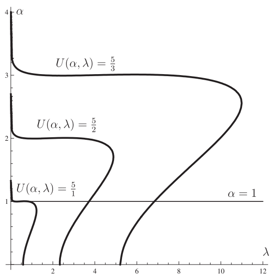

Figure 4. The curves Γ_i for $i = 1, 2, 3$.

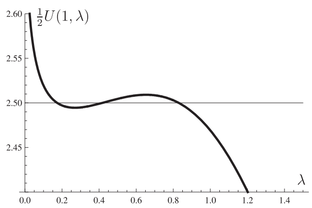

Figure 5. The graph of the function $\frac{1}{2}\,U(1,\lambda)$.

Let us consider curves $\Gamma_i:\ U(\alpha,\lambda) = \frac{5}{i}$, $(i = 1, 2, 3)$, presented in Fig. 4. A graph of function $\frac{1}{2}\,U(1,\lambda)$ has three crosspoints with straight line $y = \frac{5}{2}$ (see, Fig. 5). The abscissas $\lambda_1 = 0.17394$, $\lambda_2 = 0.417732$, $\lambda_3 = 0.829269$ relate to abscissas of crosspoints of the curve Γ_1 with straight line $\alpha = 1$.

Crosspoints (λ_1, λ_1), (λ_2, λ_2), (λ_3, λ_3) of the branch $F_1^{\pm}:\ \frac{1}{2}\,U(1,\lambda) + \frac{1}{2}\,U(1,\mu) = 5$ with the bisectrix are presented in Fig. 6. The branch consists of two components, one component is bounded. The straight line $\lambda = \lambda_0$, $\lambda_0 = 0.3$, intersects $F_1^{\pm}$ at three points (λ_0, μ_j) $(j = 1, 2, 3)$, where $\mu_1 = 0.144924$, $\mu_2 = 0.511889$, $\mu_3 = 0.772549$. Three solutions of the Neumann problem (3.1), $f = g$ in the interval $[0, 5]$ with one zero corresponding to pairs (λ_0, μ_j) $(j = 1, 2, 3)$ are depicted in Fig. 7.

The relative location of first three branches of the spectrum is shown in Fig. 8.

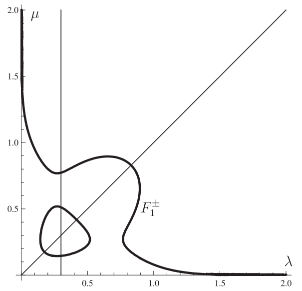

Figure 6. The branch $F_1^{\pm}$ consists of two components.

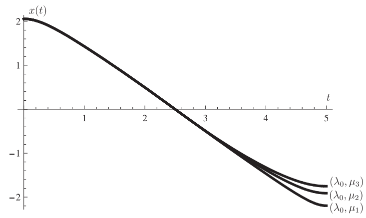

Figure 7. Solutions of the problem (3.1), $f = g$, $a = 0$, $b = 5$, for $\lambda = \lambda_0$, $\mu = \mu_1, \mu_2, \mu_3$.

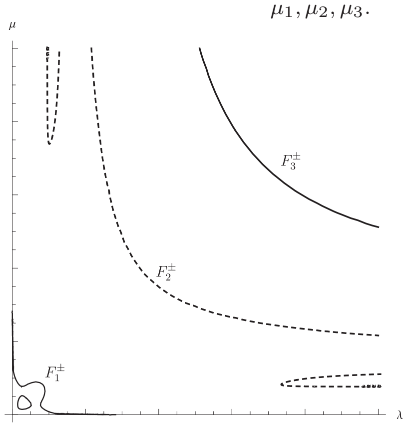

Figure 8. The first three branches of the spectrum (scheme).

5 Conclusions

We see that the spectrum of the problem with nonlinear f and g may differ essentially from the classical one. The branches of the spectrum do not intersect, and they can be located asymmetrically with respect to the bisectrix. The structure of the branches depend on properties of the functions U and V, which, in turn, are defined via t_1 and τ_1 functions. It appears that the non-standard behaviour of branches of the spectrum is due to non-monotonicity of the functions U and/or V. Branches may contain several disjoint sets which consist of curves tending to asymptotes. The interesting thing is that branches (especially the first ones) may contain separated bounded components. Investigations of this kind are important to the study of multiple solutions of nonlinear boundary value problems.

References

[1] A. Gritsans and F. Sadyrbaev. Nonlinear spectra for parameter dependent ordinary differential equations. *Nonl. Anal., Model. Control*, **12**(2):253–267, 2007.

[2] A. Gritsans and F. Sadyrbaev. On nonlinear Fučík type spectra. *Math. Model. Anal.*, **13**(2):203–210, 2008. (doi:10.3846/1392-6292.2008.13.203-210)

[3] A. Gritsans and F. Sadyrbaev. Two-parametric nonlinear eigenvalue problems. In L. Hatvani and T. Krisztin(Eds.), *Proc. of the 8'th Colloquium on the Qualitative Theory of Differential Equations, Szeged, Hungary, 2007*, pp. 1–14, 2008.

[4] P. Korman. Global solution branches and exact multiplicity of solutions for two point boundary value problems. In A. Cañada, P. Drábek and A. Fonda(Eds.), *Handbook of Differential Equations, Ordinary Differential Equations, Vol. 3*, pp. 547–606. Elsevier Science, North-Holland, Amsterdam, 2006.

[5] M. Krasnoselskii *et al. Plane vector fields.* Fizmatgiz, Moscow, 1963. (in Russian)

[6] B. Liu. On Littlewood's boundedness problem for sublinear Duffing equations. *Trans. Am. Math. Soc.*, **353**(4):1567–1585, 2000.

[7] E. Massa and B. Ruf. A global characterization of the Fučík spectrum for a system of ordinary differential equations. *J. Differ. Equations*, **234**(1):311–336, 2007.