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Some Notes on Matrix Transforms of Summability Domains of Cesàro Matrices

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Abstract. In this paper sufficient conditions for a matrix $M = (m_{nk})$ (m_{nk} are Cesàro numbers A^s_{n-k}, $s \in \mathcal{C}$ if $k \le n$ and $m_{nk} = 0$ if $k > n$) to be a transform from the summability domain of the Cesàro method C^α into the summability domain of another Cesàro method C^β, where $\alpha, \beta \in \mathcal{C}\backslash\{-1, -2, \ldots\}$, are found.

Keywords: matrix transformations, summability method of Cesàro.

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1 Introduction

Let $C^\alpha = (c_{nk})$, $\alpha \in \mathcal{C}\backslash\{-1, -2, \ldots\}$, be a series-to-sequence Cesàro method, i.e. (see [4] or [5])

$$c_{nk} = \begin{cases} \dfrac{A^\alpha_{n-k}}{A^\alpha_n} & (k \le n), \\ 0 & (k > n), \end{cases}$$

where $A^\alpha_n = \binom{n+\alpha}{n}$ are Cesàro numbers. Throughout this paper we assume that summation indices run from 0 to ∞ unless otherwise specified. A series $x := \sum x_k$ is said to be C^α-*summable* if the sequence $C^\alpha x = (C^\alpha_n x)$ is convergent, where

$$C^\alpha_n x = \sum_{k=0}^{n} c_{nk} x_k.$$

We denote the domain of all C^α-summable series by c_{C^α}, i.e.

$$c_{C^\alpha} := \Big\{ x = (x_n) \,\Big|\, \lim_{n\to\infty} C^\alpha_n x \ \text{ exists} \Big\}.$$

In [1, 3, 10] necessary and sufficient conditions for a matrix M with real or complex entries to be a transform from c_{C^α} into c_{C^β} for $\alpha, \beta \in \mathcal{R}$ or $\alpha, \beta \in \mathcal{C}\backslash\{-1, -2, \ldots\}$ are described. The summability domains (and the subsets of summability domains) of different Cesàro methods are compared in several papers (see, for example, [2, 7, 8]). For double Cesàro methods this problem has in recent years been considered, for example, in [6, 9].

In the present paper the particular subcase of the above-described problem is studied: sufficient conditions for a matrix $M = (m_{nk})$, defined by the relation

$$m_{nk} = \begin{cases} A^s_{n-k} & (k \le n,\ s \in \mathcal{C}), \\ 0 & (k > n), \end{cases} \tag{1.1}$$

to be a transform from c_{C^α} into c_{C^β}, $\alpha, \beta \in \mathcal{C}\backslash\{-1, -2, \ldots\}$ are found. It is easy to see that this problem is equivalent to the problem of finding sufficient conditions for $c_{C^\alpha} \subset c_G$, where $G := C^\beta M$.

2 Auxiliary Results

For the proof of main results we need the following properties of Cesàro numbers (see [4], p. 77–81):

$$\sum_{k=0}^{n} A^\alpha_{n-k} A^\beta_k = A^{\alpha+\beta+1}_n \text{ for every } \alpha, \beta \in \mathcal{C}, \tag{2.1}$$

$$|A^\alpha_n| \le K_1 (n+1)^{Re \alpha} \quad \text{for every } \alpha \in \mathcal{C},\ K_1 > 0, \tag{2.2}$$

$$|A^\alpha_n| \ge K_2 (n+1)^{Re \alpha} \quad \text{for } \alpha \in \mathcal{C}\backslash\{-1, -2, \ldots\},\ K_2 > 0. \tag{2.3}$$

Further we also use the following lemma.

Lemma 1. *Let $\alpha \in \mathcal{C}$, $\beta \in \mathcal{C}$. The following assertions hold:*
(A) *If $Re\, \alpha \ne -1$ and $Re\, \beta \ne -1$,* or *$\alpha = \beta = -1$, then*

$$D_n := \sum_{k=0}^{n} \left| A^\alpha_{n-k} A^\beta_k \right| = \mathcal{O}\left[(n+1)^{Re\, \alpha}\right] + \mathcal{O}\left[(n+1)^{Re\, \beta}\right] + \mathcal{O}\left[(n+1)^{Re\, (\alpha+\beta)+1}\right]. \tag{2.4}$$

(B) *If $Re\, \beta = -1$, then*

$$D_n = \begin{cases} \mathcal{O}\left[(n+1)^{Re\, \alpha} \ln(n+1)\right] & (Re\, \alpha \ge -1), \\ \mathcal{O}\left[(n+1)^{-1}\right] & (Re\, \alpha < -1). \end{cases}$$

(C) *If $Re\, \alpha = -1$, then*

$$D_n = \begin{cases} \mathcal{O}\left[(n+1)^{Re\, \beta} \ln(n+1)\right] & (Re\, \beta \ge -1), \\ \mathcal{O}\left[(n+1)^{-1}\right] & (Re\, \beta < -1). \end{cases}$$

Proof. First we note that for all α, $\beta \in \mathcal{R}$ relation (2.4) is proved, for example, in [4], p. 79–81. Let now $Re\, \alpha \ne -1$, $Re\, \beta \ne -1$. Then by (2.2) and (2.3) there exist $K_1^1, K_1^2, K_2^1, K_2^2 > 0$ so that

$$D_n \le K_1^1 K_1^2 \sum_{k=0}^{n} (n-k+1)^{Re\,\alpha}(k+1)^{Re\,\beta} \le K_1^1 K_1^2 K_2^1 K_2^2 \sum_{k=0}^{n} \left| A_{n-k}^{Re\,\alpha} A_k^{Re\,\beta} \right|.$$

Hence relation (2.4) holds, since $Re\alpha, Re\beta \in \mathcal{R}$. Thus assertion (A) is satisfied.

Let now $Re\beta = -1$. Then with the help of relation (2.2) we have

$$D_n \le M V_n, \quad M > 0,$$

where

$$V_n := \sum_{k=0}^{n} v_{nk}, \quad v_{nk} := (n-k+1)^{Re\,\alpha}(k+1)^{-1}.$$

Further we can write

$$V_n = \begin{cases} \sum\limits_{k=0}^{\frac{n}{2}-1} v_{nk} + \left(\frac{n}{2}+1\right)^{Re\,\alpha - 1} + \sum\limits_{k=\frac{n}{2}+1}^{n} v_{nk} & (n \text{ is even }), \\ \sum\limits_{k=0}^{\frac{n-1}{2}} v_{nk} + \sum\limits_{k=\frac{n+1}{2}}^{n} v_{nk} & (n \text{ is odd}), \end{cases}$$

i.e.,

$$\begin{aligned} V_n = {} & (n+1)^{Re\,\alpha} 1^{-1} + n^{Re\,\alpha} 2^{-1} + \cdots + \left(\frac{n}{2}+3\right)^{Re\,\alpha} \left(\frac{n}{2}-1\right)^{-1} \\ & + \left(\frac{n}{2}+2\right)^{Re\,\alpha} \left(\frac{n}{2}\right)^{-1} + \left(\frac{n}{2}+1\right)^{Re\,\alpha - 1} + \left(\frac{n}{2}\right)^{Re\,\alpha} \left(\frac{n}{2}+2\right)^{-1} \\ & + \cdots + 2^{Re\,\alpha} n^{-1} + 1^{Re\,\alpha}(n+1)^{-1} \end{aligned}$$

for an even number n, and

$$\begin{aligned} V_n = {} & (n+1)^{Re\,\alpha} 1^{-1} + n^{Re\,\alpha} 2^{-1} + \cdots + \left(\frac{n+5}{2}\right)^{Re\,\alpha} \left(\frac{n-1}{2}\right)^{-1} \\ & + \left(\frac{n+3}{2}\right)^{Re\,\alpha} \left(\frac{n+1}{2}\right)^{-1} + \left(\frac{n+1}{2}\right)^{Re\,\alpha} \left(\frac{n+3}{2}\right)^{-1} \\ & + \left(\frac{n-1}{2}\right)^{Re\,\alpha} \left(\frac{n+1}{2}\right)^{-1} + \cdots + 2^{Re\,\alpha} n^{-1} + 1^{Re\,\alpha}(n+1)^{-1} \end{aligned}$$

for an odd number n. Let us suppose first that $Re\,\alpha \ge -1$. Then

$$\begin{aligned} & (n+1)^{Re\,\alpha} 1^{-1} \ge 1^{Re\,\alpha}(n+1)^{-1}, \\ & n^{Re\,\alpha} 2^{-1} \ge 2^{Re\,\alpha} n^{-1}, \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & \left(\frac{n}{2}+2\right)^{Re\,\alpha} \left(\frac{n}{2}\right)^{-1} \ge \left(\frac{n}{2}\right)^{Re\,\alpha} \left(\frac{n}{2}+2\right)^{-1}, \end{aligned}$$

if n is an even number, and

$$(n+1)^{Re\,\alpha}1^{-1} \geq 1^{Re\,\alpha}(n+1)^{-1},$$
$$n^{Re\,\alpha}2^{-1} \geq 2^{Re\,\alpha}n^{-1},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\left(\frac{n+3}{2}\right)^{Re\,\alpha}\left(\frac{n+1}{2}\right)^{-1} \geq \left(\frac{n+1}{2}\right)^{Re\,\alpha}\left(\frac{n+3}{2}\right)^{-1},$$

if n is an odd number. Consequently

$$V_n \leq \begin{cases} 2\sum_{k=0}^{\frac{n}{2}-1}(n-k+1)^{Re\,\alpha}(k+1)^{-1} + \left(\frac{n}{2}+1\right)^{Re\,\alpha-1} & (n \text{ is even}), \\ 2\sum_{k=0}^{\frac{n-1}{2}}(n-k+1)^{Re\,\alpha}(k+1)^{-1} & (n \text{ is odd}). \end{cases}$$

Therefore

$$V_n \leq \begin{cases} 2\left(\frac{n}{2}+2\right)^{Re\,\alpha}\sum_{k=0}^{\frac{n}{2}-1}(k+1)^{-1} + \left(\frac{n}{2}+1\right)^{Re\,\alpha-1} & (n \text{ is even}), \\ 2\left(\frac{n+3}{2}\right)^{Re\,\alpha}\sum_{k=0}^{\frac{n-1}{2}}(k+1)^{-1} & (n \text{ is odd}) \end{cases}$$

if $-1 \leq Re\,\alpha \leq 0$, and

$$V_n \leq \begin{cases} 2(n+1)^{Re\,\alpha}\sum_{k=0}^{\frac{n}{2}-1}(k+1)^{-1} + \left(\frac{n}{2}+1\right)^{Re\,\alpha-1} & (n \text{ is even}), \\ 2(n+1)^{Re\,\alpha}\sum_{k=0}^{\frac{n-1}{2}}(k+1)^{-1} & (n \text{ is odd}) \end{cases}$$

if $Re\,\alpha > 0$. Hence

$$D_n = \mathcal{O}\left[(n+1)^{Re\,\alpha}\ln(n+1)\right];\ Re\,\alpha \geq -1. \tag{2.5}$$

We assume now that $Re\,\alpha < -1$. Then

$$(n+1)^{Re\,\alpha}1^{-1} \leq 1^{Re\,\alpha}(n+1)^{-1},$$
$$n^{Re\,\alpha}2^{-1} \leq 2^{Re\,\alpha}n^{-1},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\left(\frac{n}{2}+2\right)^{Re\,\alpha}\left(\frac{n}{2}\right)^{-1} \leq \left(\frac{n}{2}\right)^{Re\,\alpha}\left(\frac{n}{2}+2\right)^{-1},$$

if n is an even number, and

$$(n+1)^{Re\,\alpha}1^{-1} \leq 1^{Re\,\alpha}(n+1)^{-1},$$
$$n^{Re\,\alpha}2^{-1} \leq 2^{Re\,\alpha}n^{-1},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\left(\frac{n+3}{2}\right)^{Re\,\alpha}\left(\frac{n+1}{2}\right)^{-1} \leq \left(\frac{n+1}{2}\right)^{Re\,\alpha}\left(\frac{n+3}{2}\right)^{-1},$$

if n is an odd number. Consequently

$$V_n \leq \begin{cases} 2\sum\limits_{k=\frac{n}{2}+1}^{n}(n-k+1)^{Re\,\alpha}(k+1)^{-1}+\left(\frac{n}{2}+1\right)^{Re\,\alpha-1} & (n \text{ is even}),\\ 2\sum\limits_{k=\frac{n+1}{2}}^{n}(n-k+1)^{Re\,\alpha}(k+1)^{-1} & (n \text{ is odd}).\end{cases}$$

Therefore

$$V_n \leq \begin{cases} 4(n+4)^{-1}\sum\limits_{k=\frac{n}{2}+1}^{n}(n-k+1)^{Re\,\alpha}+\left(\frac{n}{2}+1\right)^{Re\,\alpha-1} & (n \text{ is even}),\\ 4(n+3)^{-1}\sum\limits_{k=\frac{n+1}{2}}^{n}(n-k+1)^{Re\,\alpha} & (n \text{ is odd}).\end{cases}$$

Hence

$$D_n = \mathcal{O}\left[(n+1)^{-1}\right];\ Re\,\alpha < -1). \tag{2.6}$$

Consequently assertion (B) holds by (2.5) and (2.6). The proof of assertion (C) is similar to the proof of assertion (B). So we omit it. □

3 Main Results

Now we are able to prove the main result of this paper.

Theorem 1. *Let $\alpha, \beta \in \mathcal{C}\backslash\{-1,-2,\ldots\}$ and $s \in \mathcal{C}$. If $Re\,s < -1$ and $Re\,s < Re\,\alpha \leq Re\,\beta$, then the matrix $M = (m_{nk})$, defined by relation* (1.1), *transforms c_{C^α} into c_{C^β}.*

Proof. It is sufficient to show that $c_{C^\alpha} \subset c_G$, where $G = C^\beta M := (g_{nk})$. Using equality (2.1), we get

$$g_{nl} = \frac{1}{A_n^\beta}\sum_{k=l}^{n} A_{n-k}^\beta A_{k-l}^s = \frac{1}{A_n^\beta}\sum_{k=0}^{n-l} A_{n-l-k}^\beta A_k^s = \frac{A_{n-l}^{\beta+s+1}}{A_n^\beta}.$$

As the inverse matrix (η_{lk}) of $C^\alpha = (c_{nk})$ is defined by the equalities (see [4], p. 86)

$$\eta_{lk} = \begin{cases} A_k^\alpha A_{l-k}^{-\alpha-2} & (k \leq l),\\ 0 & (k > l),\end{cases}$$

for every $x = (x_k) \in c_{C^\alpha}$ we get

$$\sum_{k=0}^{n} g_{nk}x_k = \sum_{k=0}^{n}\gamma_{nk}y_k,$$

where $y_k := C_k^\alpha x$ and $\gamma_{nk} = A_{n-k}^{\beta+s-\alpha}A_k^\alpha / A_n^\beta$ by equality (2.1). Consequently, for $c_{C^\alpha} \subset c_G$ it is sufficient to show by the well-known theorem of Kojima-Schur that

$$\textit{there exists the finite limits}\ \lim{}_n\gamma_{nk},\ \lim{}_n\sum_{k=0}^{n}\gamma_{nk}, \tag{3.1}$$

and

$$\sum_k |\gamma_{nk}| = \mathcal{O}(1), \tag{3.2}$$

since the sequence (y_k) is convergent for every $x \in c_{C^\alpha}$. Thus, with the help of relations (2.1)–(2.3) we have

$$\Big|\sum_{k=0}^{n} \gamma_{nk}\Big| = \Big|\frac{A_n^{\beta+s+1}}{A_n^{\beta}}\Big| = \mathcal{O}(1)(n+1)^{Re\, s+1} = o(1)$$

(since $Re\, s + 1 < 0$), and

$$\begin{aligned} |\gamma_{nk}| &= \mathcal{O}(1)\frac{(n-k+1)^{Re\,(\beta+s-\alpha)}}{(n+1)^{Re\,\beta}} \\ &= \mathcal{O}(1)\left(1-\frac{k}{n+1}\right)^{Re\,\beta}(n-k+1)^{Re\,(s-\alpha)} = o(1) \end{aligned}$$

(since $Re\,(s-\alpha) < 0$). Thus condition (3.1) is fulfilled.

The proof of validity of condition (3.2) we divide into three parts.

1) Let $Re\,(\beta+s-\alpha) \neq -1$, $Re\,\alpha \neq -1$, or $\beta+s-\alpha = \alpha = -1$. Then we get

$$\begin{aligned} S_n := \sum_{k=0}^{n} \Big|A_{n-k}^{\beta+s-\alpha} A_k^{\alpha}\Big| = \mathcal{O}\Big[(n+1)^{Re\,(\beta+s-\alpha)}\Big] + \mathcal{O}\Big[(n+1)^{Re\,\alpha}\Big] \\ + \mathcal{O}\Big[(n+1)^{Re\,(\beta+s)+1}\Big] \end{aligned}$$

by Lemma 1. If

$$L := \max\Big\{Re\,(\beta+s-\alpha), Re\,\alpha, Re\,(\beta+s)+1\Big\} = Re\,\big(\beta+s-\alpha\big),$$

then $S_n = \mathcal{O}\left[(n+1)^{Re\,(\beta+s-\alpha)}\right]$, and consequently with the help of (2.3) we have

$$T_n := \sum_{k=0}^{n} \Big|\gamma_{nk}\Big| = \frac{S_n}{|A_n^{\beta}|} = \mathcal{O}\left[(n+1)^{Re\,(s-\alpha)}\right] = \mathcal{O}(1).$$

If $L = Re\,(\beta+s)+1$, then using (2.3) we can conclude that

$$S_n = \mathcal{O}\left[(n+1)^{Re\,(\beta+s)+1}\right],$$

and therefore

$$T_n = \mathcal{O}\left[(n+1)^{Re\,s+1}\right] = \mathcal{O}(1).$$

If $L = Re\,\alpha$, then $S_n = \mathcal{O}\left[(n+1)^{Re\,\alpha}\right]$, and hence

$$T_n = \mathcal{O}\left[(n+1)^{Re\,(\alpha-\beta)}\right] = \mathcal{O}(1),$$

i.e. condition (3.2) holds.

2) Let $Re\,\alpha = -1$. Then

$$S_n = \begin{cases} \mathcal{O}\left[(n+1)^{Re\,(\beta+s-\alpha)}\ln(n+1)\right] & (Re\,(\beta+s-\alpha) \geq -1), \\ \mathcal{O}\left[(n+1)^{-1}\right] & (Re\,(\beta+s-\alpha) < -1), \end{cases}$$

and consequently

$$T_n = \begin{cases} \mathcal{O}\left[(n+1)^{Re\,(s-\alpha)}\ln(n+1)\right] & (Re\,(\beta+s-\alpha) \geq -1), \\ \mathcal{O}\left[(n+1)^{-Re\,\beta-1}\right] & (Re\,(\beta+s-\alpha) < -1). \end{cases}$$

Therefore $T_n = \mathcal{O}(1)$, because $Re\,(s-\alpha) < 0$ and $Re\,\beta \geq Re\,\alpha = -1$, i.e. condition (3.2) holds.

3) Let $Re\,(\beta+s-\alpha) = -1$. Then

$$S_n = \begin{cases} \mathcal{O}\left[(n+1)^{Re\,\alpha}\ln(n+1)\right] & (Re\,\alpha \geq -1), \\ \mathcal{O}\left[(n+1)^{-1}\right] & (Re\,\alpha < -1), \end{cases}$$

and consequently

$$T_n = \begin{cases} \mathcal{O}\left[(n+1)^{Re\,(\alpha-\beta)}\ln(n+1)\right] & (Re\,\alpha \geq -1), \\ \mathcal{O}\left[(n+1)^{-Re\,\beta-1}\right] & (Re\,\alpha < -1). \end{cases}$$

Therefore $T_n = \mathcal{O}(1)$, because $Re\,(\alpha-\beta) = Re\,s + 1 < 0$ and $-Re\,\beta - 1 = Re\,(s-\alpha) < 0$, i.e. condition (3.2) holds. □

It is well known that C^β includes C^α, i.e. $c_{C^\beta} \supseteq c_{C^\alpha}$, if $Re\,\beta > Re\,\alpha > -1$ (see [4], p. 87). Therefore for real numbers α, β we get

Corollary 1. Let $\alpha, \beta \in \mathcal{R}$, $s \in \mathcal{C}$ with $\alpha, \beta > -1$, $Re\,s < -1$ and M be defined by (1.1). If M transforms c_{C^α} into c_{C^β}, then C^β includes C^α.

Proof. We see from the proof of Theorem 1 that the validity of condition (3.2) is necessary for M to be a transformation from c_{C^α} into c_{C^β}. We prove that condition (3.2) is not satisfied for $\beta < \alpha$. Indeed, by (2.2) and (2.3) there exists a number $K > 0$ so that

$$T_n = \sum_{k=0}^{n}\left|\frac{A_{n-k}^{\beta+s-\alpha}A_k^\alpha}{A_n^\beta}\right| = \sum_{k=0}^{n-1}\left|\frac{A_{n-k}^{\beta+s-\alpha}A_k^\alpha}{A_n^\beta}\right| + \left|\frac{A_n^\alpha}{A_n^\beta}\right| \geq \left|\frac{A_n^\alpha}{A_n^\beta}\right| \geq K\left[(n+1)^{Re\,(\alpha-\beta)}\right].$$

As the sequence $\left((n+1)^{Re\,(\alpha-\beta)}\right)$ is not bounded for $\beta < \alpha$, then also the sequence (T_n) is not bounded for $\beta < \alpha$. Consequently for the validity of condition (3.2) it is necessary that $\beta \geq \alpha$. As $\alpha, \beta > -1$, then C^β includes C^α. □

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