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Periodic Waves in Microstructured Solids and Inverse Problems*

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Abstract. Conditions for the existence of periodic and solitary waves in 1D microstructured solid of Mindlin type are deduced. Inverse problems to determine material properties are solved.

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AMS Subject Classification: 74J30; 74J35; 74E99; 74J25; 35R30.

1 Introduction

Microstructured materials like alloys, crystallites, ceramics, functionally graded materials, etc. are gaining a growing importance in contemporary high technology [4, 9, 13, 16, 20]. This brings along a growing necessity for modelling mechanical processes in these materials and non-destructive evaluation of physical parameters.

There are several models of microstructure (see e.g. [5]), but in this paper we follow the model that was posed by Mindlin [15]. This model was later adjusted to Rayleigh waves [7] (see also related works [6] and [3]) and approximated by a Boussinesq-type equation [4]. In the linear case the Rayleigh waves as well as packets of harmonic waves are informative in the sense of the inverse problems to determine physical parameters [7, 9, 10, 11].

In the nonlinear case, under a certain balance of nonlinearity and dispersion, solitons may emerge. Existence of solitary waves in the one-dimensional case was proved both for the Boussinesq-type approximation and an original Mindlin's coupled system [12, 14]. Numerical simulations [2, 17, 21] and physical observations [18, 19] support the theoretical results. Solitary waves can be used to reconstruct material parameters [8, 12].

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In this paper, for the first time periodic waves in nonlinear microstructured materials of Mindlin's type are studied from the theoretical viewpoint. The study is limited to the 1D case. The existence conditions of solitary waves, deduced formerly in [12], follow as certain limits of the existence conditions of periodic waves. In the last part of the paper we solve inverse problems to reconstruct material parameters by means of measurements of periodic and solitary waves.

2 Description of Mathematical Model and Derivation of Equations

The mathematical model of the microstructured solid of Mindlin type is based on assumptions that the microstructure can be interpreted as deformable cells like "a molecule of polymer or a crystallite of a polycrystal". It leads to the necessity to consider deformations on two scales, on macro - and microscopic scales [15]. The macrodisplacement u is defined as a usual displacement of a material particle by its components $u_i = x_i - X_i$, where x_i and X_i are the components of the spatial and material position vectors, respectively. Analogically the microdisplacement is defined by $u'_i = x'_i - X'_i$, where the origin of coordinates x'_i is inside the cell and moves with the displacement u. The microdisplacement is assumed to be linearly dependent on microcoordinates, i.e. $u'_i = x'_j \psi_{ji}(x_i, t)$, where ψ_{ji} is the microdeformation tensor [4, 17]. In this paper we will consider the 1D case so the indices i, j will be omitted, i.e. $u_1 = u$ and $\psi_{11} = \psi$. We follow the technique introduced in [4]. The governing equations of 1D model are

$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial \sigma}{\partial x}, \\ I \dfrac{\partial^2 \psi}{\partial t^2} = \dfrac{\partial \tau}{\partial x} - \kappa, \end{cases}$$

where

$$\sigma := \frac{\partial W}{\partial u_x}, \quad \tau := \frac{\partial W}{\partial \psi_x}, \quad \kappa := \frac{\partial W}{\partial \psi},$$

ρ is the macro-density, I is the micro-inertia and W is the free energy function that is assumed to have the form

$$W = \frac{A u_x^2}{2} + \frac{B \psi_x^2}{2} + \frac{C \psi^2}{2} + D u_x \psi + \frac{E u_x^3}{6} + \frac{F \psi_x^3}{6}.$$

Here $A, \ldots, F$ are constants and subscripts here and below denote differentiation. Then the governing equations take the form

$$\begin{cases} \rho u_{tt} = A u_{xx} + D \psi_x + \frac{E}{2} \left(u_x^2 \right)_x, \\ I \psi_{tt} = B \psi_{xx} + \frac{F}{2} \left(\psi_x^2 \right)_x - D u_x - C \psi. \end{cases}$$

Now we introduce nondimensional variables X, T, U and the quantities I^*, B^* and F^* by means of the following relations:

$$x = LX, \quad u = \epsilon L U, \quad t = \frac{LT}{c_0}, \quad I = I^* l^2, \quad F = F^* l^3, \quad B = B^* l^2,$$

where L and ϵL are fixed lengths (possibly a length and an amplitude of a wave), c_0 is a fixed velocity and l is a size of the microstructure. By using these relations we obtain the following nondimensional equations:

$$\frac{\rho c_0^2 \epsilon L}{L^2} U_{TT} = \frac{A\epsilon L}{L^2} U_{XX} + \frac{D}{L}\psi_X + \frac{E\epsilon^2 L^2}{2L^3}\left(U_X^2\right)_X,$$
$$\frac{I^* l^2 c_0^2}{L^2}\psi_{TT} = \frac{B^* l^2}{L^2}\psi_{XX} + \frac{F^* l^3}{2L^3}\left(\psi_X^2\right)_X - \frac{D\epsilon L}{L} U_X - C\psi.$$

Let us divide the first equation by $\frac{\rho c_0^2 \epsilon L}{L^2}$ and the second by $I^* c_0^2$. Also we differentiate the first equation with respect to X and introduce a new variable $v = U_X$ and a scale coefficient $\delta = \frac{l^2}{L^2}$. Having done that, we reach the following equations:

$$v_{TT} = \frac{A}{\rho c_0^2} v_{XX} + \frac{D}{\rho c_0^2 \epsilon}\psi_{XX} + \frac{E\epsilon}{2\rho c_0^2}\left(v^2\right)_{XX},$$
$$\delta\psi_{TT} = \delta\frac{B^*}{I^* c_0^2}\psi_{XX} + \delta^{3/2}\frac{F^*}{2I^* c_0^2}\left(\psi_X^2\right)_X - \frac{D\epsilon}{I^* c_0^2} v - \frac{C}{I^* c_0^2}\psi.$$

For simplicity we use lowercase letters x and t instead of X and T and define new coefficients:

$$a_0 = \frac{A}{\rho c_0^2},\ \alpha = \frac{D}{\rho c_0^2 \epsilon},\ \mu = \frac{E\epsilon}{\rho c_0^2},\ a_1 = \frac{B^*}{I^* c_0^2},\ \lambda = \frac{F^*}{I^* c_0^2},\ \gamma = \frac{C}{I^* c_0^2},\ \beta = \frac{D\epsilon}{I^* c_0^2}.$$

Then the system of the equations is transformed to

$$\begin{cases} v_{tt} = a_0 v_{xx} + \alpha\psi_{xx} + \frac{\mu}{2}\left(v^2\right)_{xx}, \\ \delta\psi_{tt} = \delta a_1\psi_{xx} + \delta^{3/2}\lambda\psi_x\psi_{xx} - \gamma\psi - \beta v. \end{cases} \tag{2.1}$$

Here, according to the physical background, the coefficients satisfy the following conditions:

$$a_0, a_1, \delta, \gamma, \alpha\beta > 0. \tag{2.2}$$

The system is supposed to be hyperbolic, i.e. the following condition must be fulfilled by the physical parameters:

$$\gamma a_0 - \alpha\beta > 0. \tag{2.3}$$

We emphasize that in the linear case the type of dispersion is related to the sign of $\gamma a_0 - \gamma a_1 - \alpha\beta$. Namely, it holds (see [10, 12])

1. for normal dispersion: $\gamma a_0 - \gamma a_1 - \alpha\beta > 0$,

2. for dispersionless case: $\gamma a_0 - \gamma a_1 - \alpha\beta = 0$,

3. for anomalous dispersion: $\gamma a_0 - \gamma a_1 - \alpha\beta < 0$.

From now on, let us consider the travelling wave solutions. This means that the solution of the system (2.1) has the form

$$\begin{cases} v(x,t) = w(x - c_1\, t), \\ \psi(x,t) = \varphi(x - c_2\, t), \end{cases} \tag{2.4}$$

where w and φ are arbitrary functions and c_1 and c_2 are velocities of the components of the wave. In physically reasonable cases it holds $c := c_1 = c_2$. Indeed, if either w or φ is a constant, then we may set $c_1 = c_2$ without restriction of generality. But if both w and φ are non-constant, then we plug (2.4) into the first equation of (2.1) to obtain the relation $f_1(x - c_1 t) = f_2(x - c_2 t)$, where the functions $f_1 = (c_1^2 - a_0)w'' - \frac{\mu}{2}(w^2)''$ and $f_2 = \alpha\varphi''$ are not constant in physically relevant cases. From this relation we infer $c_1 = c_2$.

Thus, let us rewrite the system (2.1) in a new variable $\eta = x - ct$:

$$\begin{cases} c^2 w''(\eta) = a_0 w''(\eta) + \frac{\mu}{2}\left[w(\eta)^2\right]'' + \alpha\varphi''(\eta), \\ \delta c^2 \varphi''(\eta) = \delta a_1 \varphi''(\eta) + \delta^{3/2}\lambda\varphi'(\eta)\varphi''(\eta) - \gamma\varphi(\eta) - \beta w(\eta), \end{cases} \tag{2.5}$$

where the first equation is twice integrable. After integrating we have

$$\varphi(\eta) = \frac{1}{\alpha}\left[\left(c^2 - a_0\right)w(\eta) - \frac{\mu}{2}w(\eta)^2\right] + C_1\eta + C_0, \tag{2.6}$$

where C_0 and C_1 are constants.

We are interested in *periodic* and *solitary wave* solutions. This immediately implies $C_1 = 0$. Moreover, for the sake of simplicity we will be limited to the case $C_0 = 0$. The latter condition is always satisfied for the solitary wave solutions. But in the case of periodic w, φ and $\mu = 0$ the condition $C_0 = 0$ implies that the integrals of w, φ that are the macro- and microdeformation, respectively, are also periodic. Consequently, (2.6) takes the form

$$\varphi(\eta) = \frac{1}{\alpha}\left[\left(c^2 - a_0\right)w(\eta) - \frac{\mu}{2}w(\eta)^2\right]. \tag{2.7}$$

After replacing $\varphi(\eta)$ in the second equation in (2.5) by (2.7) the following equation for unknown function $w(\eta)$ is deduced:

$$w'' = \frac{\mu(w')^2 - \frac{\delta^{1/2}\lambda\mu(c^2-a_0)(w')^3}{\alpha(c^2-a_1)} + \frac{\delta^{1/2}\lambda\mu^2(w')^3 w}{\alpha(c^2-a_1)} + \frac{\gamma\mu w^2}{2\delta(c^2-a_1)} - \frac{\gamma(c^2-a_0)+\alpha\beta}{\delta(c^2-a_1)}w}{(c^2 - a_0 - \mu w)\left(1 - \frac{\delta^{1/2}\lambda w'}{\alpha(c^2-a_1)}(c^2 - a_0 - \mu w)\right)}. \tag{2.8}$$

Doing the substitution $y(w) = (c^2 - a_0 - \mu w(\eta))w'(\eta)$, the equation (2.8) is transformed into the more simple and integrable one:

$$\begin{aligned} &\left(\delta\left(c^2 - a_1\right) - \frac{\delta^{3/2}\lambda}{\alpha}y(w)\right)y(w)y'(w) \\ &\quad = \left(\frac{\mu\gamma}{2}w^2 - \left(\gamma\left(c^2 - a_0\right) + \alpha\beta\right)w\right)\left(c^2 - a_0 - \mu w\right). \end{aligned} \tag{2.9}$$

3 Periodic Waves

For the further analysis it is more comfortable to rewrite the equation (2.8) into the following system for the pair $(w(\eta), z(\eta))$:

$$\begin{cases} w' = z, \\ z' = \dfrac{\mu z^2 - \frac{\delta^{1/2}\lambda\mu(c^2-a_0)z^3}{\alpha(c^2-a_1)} + \frac{\delta^{1/2}\lambda\mu^2 z^3 w}{\alpha(c^2-a_1)} + \frac{\gamma\mu w^2}{2\delta(c^2-a_1)} - \frac{\gamma(c^2-a_0)+\alpha\beta}{\delta(c^2-a_1)}w}{(c^2-a_0-\mu w)\left(1-\frac{\delta^{1/2}\lambda z}{\alpha(c^2-a_1)}(c^2-a_0-\mu w)\right)}. \end{cases} \tag{3.1}$$

Depending on μ, this system has a different number of equilibrium points, i.e. the points where $z'(\eta) = w'(\eta) = 0$. If $\mu \neq 0$ then there are two equilibrium points and if $\mu = 0$ then there is a single equilibrium point. Combining two parameters of nonlinearity three different cases can be coped with:

$$1)\ \lambda = 0 \ \text{ and } \ \mu = 0, \quad 2)\ \lambda \neq 0 \ \text{ and } \ \mu = 0, \quad 3)\ \mu \neq 0.$$

The first case is purely linear and involves sinusoidal periodic travelling wave solutions [12]. We will present a more detailed treatment of the second case $\lambda \neq 0$, $\mu = 0$ in the next subsection where we deduce conditions for the velocity c that are necessary and sufficient for the existence of periodic travelling wave solutions. Results in the third case $\mu \neq 0$ can be obtained in a similar manner and will be given more shortly. Moreover, in the latter case we can present a common treatment for the subcases $\lambda = 0$ and $\lambda \neq 0$.

3.1 Case $\lambda \neq 0$ and $\mu = 0$

In this case we have the following system of nonlinear differential equations:

$$\begin{cases} w' = z, \\ z' = -\dfrac{\gamma(c^2-a_0)+\alpha\beta}{\delta(c^2-a_1)(c^2-a_0)}w \Big/ \left(1 - \dfrac{\delta^{1/2}\lambda(c^2-a_0)}{\alpha(c^2-a_1)}z\right). \end{cases} \tag{3.2}$$

The system linearized near the equilibrium point $(w; z) = (0; 0)$ is

$$\begin{cases} w' = z, \\ z' = -\dfrac{\gamma(c^2-a_0)+\alpha\beta}{\delta(c^2-a_1)(c^2-a_0)}w. \end{cases}$$

The characteristic equation of the latter system is

$$k^2 + \frac{\gamma(c^2-a_0)+\alpha\beta}{\delta(c^2-a_1)(c^2-a_0)} = 0.$$

Periodic waves are related to complex roots. Therefore, the inequality

$$\frac{\gamma(c^2-a_0)+\alpha\beta}{(c^2-a_1)(c^2-a_0)} < 0$$

must hold. Taking this into account we deduce the following conditions for the velocity c:

1. for normal dispersion, i.e., $\gamma a_0 - \gamma a_1 - \alpha\beta > 0$ and

$$\text{either } a)\ c^2 > a_0 > a_0 - \frac{\alpha\beta}{\gamma} > a_1, \text{ or } b)\ a_0 - \frac{\alpha\beta}{\gamma} > c^2 > a_1; \quad (3.3)$$

2. for dispersionless case, i.e., $\gamma a_0 - \gamma a_1 - \alpha\beta = 0$,

$$c^2 > a_0; \quad (3.4)$$

3. for anomalous dispersion, i.e., $\gamma a_0 - \gamma a_1 - \alpha\beta < 0$ and

$$\text{either } a)\ c^2 > a_0,\ a_1 > a_0 - \frac{\alpha\beta}{\gamma}, \text{ or } b)\ a_0, a_1 > c^2 > a_0 - \frac{\alpha\beta}{\gamma}. \quad (3.5)$$

Observing the second equation of the system (3.2) we see that there exists a singular line $z = z_s$ with $z_s = \frac{\alpha(c^2-a_1)}{\delta^{1/2}\lambda(c^2-a_0)}$ on the phase plane. Therefore, the first derivative of the function $w(\eta)$ is bounded by z_s. More precisely, $w' < z_s$ in the case of positive z_s and $w' > z_s$ in the case of negative z_s. Using such a restriction for w' we can deduce certain bounds for the extrema w_{min} and w_{max} of the periodic wave, too. Let us do that.

Firstly, we mention that $w_{min} = -w_{max}$. This follows from the symmetry with respect to the z-axis of the phase curves of (3.2). Further, we consider the equation (2.9) that was deduced by means of the substitution $y(w) = (c^2 - a_0)w'(\eta)$ (recall that $\mu = 0$ in the present case). Since the phase curve $z = z(w)$ has two branches, passing through the upper and lower half-planes, respectively, the function $y(w)$ also has two branches: a positive and a negative one. Let us choose such a branch of $y(w)$ that satisfies the condition $\operatorname{sign} \frac{y}{c^2-a_0} = \operatorname{sign} z_s$. Integrating (2.9) we get

$$\left(\frac{\delta(c^2-a_1)}{2} - \frac{\delta^{3/2}\lambda}{3\alpha} y\right) y^2 \Bigg|_{y_1}^{y_2} = -\frac{(c^2-a_0)(\gamma(c^2-a_0)+\alpha\beta)}{2} w^2 \Bigg|_{w_1}^{w_2}, \quad (3.6)$$

where $y_1 = y(w_1)$ and $y_2 = y(w_2)$. Setting $w_2 = w_{ext}$, where w_{ext} is either w_{max} or w_{min}, we have $y_2 = y(w_{ext}) = 0$, because $w' = 0$ in the extremum point. Moreover, let $w_1 = 0$. Then (3.6) yields

$$\left(\frac{\delta(c^2-a_1)}{2} - \frac{\delta^{3/2}\lambda}{3\alpha} y_1\right) y_1^2 = \frac{(c^2-a_0)(\gamma(c^2-a_0)+\alpha\beta)}{2} w_{ext}^2$$

with $y_1 = y(0)$. This in turn implies $w_{ext}^2 = f(z_1)$ with

$$f(z) = \frac{2(c^2-a_0)}{\gamma(c^2-a_0)+\alpha\beta} \left(\frac{\delta(c^2-a_1)}{2} - \frac{\delta^{3/2}\lambda(c^2-a_0)}{3\alpha} z\right) z^2,$$

where $z_1 = y_1/(c^2 - a_0)$. Due to the choice of the branch of y, the numbers z_1 and z_s have the same signs. Moreover, $0 < |z_1| < |z_s|$. One can immediately check that the maximum of the cubic function $f(z)$ between 0 and z_s is achieved at $z = z_s$. Therefore, $w^2_{ext} < f(z_s)$. Computing $f(z_s)$ we reach the following bound for the extrema:

$$w^2_{max} = w^2_{min} < \frac{\alpha^2(c^2 - a_1)^3}{3\lambda^2(c^2 - a_0)(\gamma(c^2 - a_0) + \alpha\beta)}. \tag{3.7}$$

Summing up, the conditions (3.3)–(3.5) give ranges of the velocity when periodic wave may exist. The inequality (3.7) shows that the amplitude of the periodic wave is restricted. In this connection, the crucial role has the microstructure nonlinearity parameter λ. The bigger λ, the smaller range of the amplitude.

3.2 Case $\mu \neq 0$

As it was mentioned, in the case $\mu \neq 0$ two equilibrium points occur. Let us consider separately these two cases.

1. Waves related to the equilibrium point $(w; z) = (0; 0)$.

Using the same technique of linearisation near the equilibrium points as in Subsection 3.1 we deduce the following restrictions for the velocity c:

1. for normal dispersion, i.e., $\gamma a_0 - \gamma a_1 - \alpha\beta > 0$ and

$$\text{either } a)\ c^2 > a_0 > a_0 - \frac{\alpha\beta}{\gamma} > a_1 \quad \text{or} \quad b)\ a_0 - \frac{\alpha\beta}{\gamma} > c^2 > a_1; \tag{3.8}$$

2. for dispersionless case, i.e., $\gamma a_0 - \gamma a_1 - \alpha\beta = 0$,

$$c^2 > a_0; \tag{3.9}$$

3. for anomalous dispersion, i.e., $\gamma a_0 - \gamma a_1 - \alpha\beta < 0$ and

$$\text{either } a)\ c^2 > a_0, a_1 > a_0 - \frac{\alpha\beta}{\gamma} \quad \text{or } b)\ a_0, a_1 > c^2 > a_0 - \frac{\alpha\beta}{\gamma}. \tag{3.10}$$

Observing the second factor of the denominator of the second equation in the system (3.1) we see that the variable $y(w) = (c^2 - a_0 - \mu w(\eta))w'(\eta)$ has the singular value $y_s = \frac{\alpha(c^2-a_1)}{\delta^{1/2}\lambda}$. Therefore, the value of y is located between 0 and y_s. This enables to deduce estimates for the extrema. Let us integrate the equation (2.9):

$$\begin{aligned} &\left(\frac{\delta(c^2 - a_1)}{2} - \frac{\delta^{3/2}\lambda}{3\alpha} y\right) y^2 \Big|_{y_1}^{y_2} \\ &= \left[-\frac{\mu^2\gamma}{8} w^4 + \frac{\mu(3\gamma(c^2 - a_0) + 2\alpha\beta)}{6} w^3 - \frac{(c^2 - a_0)(\gamma(c^2 - a_0) + \alpha\beta)}{2} w^2\right]\Big|_{w_1}^{w_2}, \end{aligned} \tag{3.11}$$

where $y_1 = y(w_1)$ and $y_2 = y(w_2)$. The present situation is more complicated than in the case $\mu = 0$, because the right- hand side of (3.11) contains a quartic polynomial instead of the simple square of w. Therefore, in the first stage we will get results for such a polynomial of w_{ext}, not directly for w_{ext}.

Let us set $w_2 = w_{ext}$, where w_{ext} is again either w_{max} or w_{min}, and $w_1 = 0$, $y_1 = y(0)$ in (3.11) to get

$$\left(\frac{\delta(c^2-a_1)}{2} - \frac{\delta^{3/2}\lambda}{3\alpha}y_1\right)y_1^2 = R_1 w_{ext}^4 + R_2 w_{ext}^3 + R_3 w_{ext}^2, \tag{3.12}$$

where

$$R_1 = \frac{\gamma\mu^2}{8},\ R_2 = -\frac{3\mu\gamma(c^2-a_0)+2\mu\alpha\beta}{6},\ R_3 = \frac{(c^2-a_0)(\gamma(c^2-a_0)+\alpha\beta)}{2}.$$

Arguing similarly as in Subsection 3.1 we deduce the following estimate for the extrema in an implicit form:

$$\lambda^2 < \frac{R_0}{R_1 w_{max}^4 + R_2 w_{max}^3 + R_3 w_{max}^2} = \frac{R_0}{R_1 w_{min}^4 + R_2 w_{min}^3 + R_3 w_{min}^2}, \tag{3.13}$$

where

$$R_0 = \alpha^2(c^2-a_1)^3/6. \tag{3.14}$$

Moreover, from (3.12) another useful equation follows:

$$R_1\left(w_{max}^4 - w_{min}^4\right) + R_2\left(w_{max}^3 - w_{min}^3\right) + R_3\left(w_{max}^2 - w_{min}^2\right) = 0. \tag{3.15}$$

To deduce explicit bounds for the extrema of w, we make use of the second singular value of w $\frac{c^2-a_0}{\mu}$ of the system (3.1) and the circumstance that w cannot reach the second equilibrium point $\frac{2}{\mu}(c^2-a_0+\frac{\alpha\beta}{\gamma})$ in the case of periodic wave. We can argue as follows. Due to the Cauchy's theorem, the solution of (3.1) is unique for a given initial condition. This implies that the phase curves related to different periodic waves cannot intersect and they form a family of closed curves inserted into each-other. Consequently, decreasing w_{min} results in the increase of the corresponding value of w_{max} and vice versa. If we approach with w_{min} or w_{max} any of the bounds $\frac{c^2-a_0}{\mu}$ or $\frac{2}{\mu}(c^2-a_0+\frac{\alpha\beta}{\gamma})$, the corresponding bound for the opposite extremum can be found solving the equation (3.15).

Let us consider in detail the particular case $\frac{2}{\mu}(c^2-a_0+\frac{\alpha\beta}{\gamma}) \le \frac{c^2-a_0}{\mu} < 0$ and $\mu > 0$. Then $0 < c^2 \le a_0 - \frac{2\alpha\beta}{\gamma}$ and the lower bound of w_{min} is $\frac{c^2-a_0}{\mu}$. We plug this bound for w_{min} into (3.15) and deduce the following equation for the limit of the amplitude $A = w_{max} - w_{min}$:

$$\frac{A^2}{24}\left(3\gamma\mu^2A^2 - 8\mu\alpha\beta A - 6\left(c^2-a_0\right)\left(2\alpha\beta - \gamma a_0 + \gamma c^2\right)\right) = 0.$$

The positive nontrivial solution is

$$A_{ext} = \frac{1}{6\gamma\mu}\left[8\alpha\beta + \sqrt{72\gamma^2\left(c^2-a_0+\frac{4\alpha\beta}{\gamma}\right)\left(c^2-a_0+\frac{2\alpha\beta}{3\gamma}\right)}\right].$$

Therefore, we obtain $\frac{c^2-a_0}{\mu} < w_{min} < w_{max} < \frac{c^2-a_0}{\mu} + A_{ext}$.

Having studied all cases of location of $\frac{c^2-a_0}{\mu}$ and $\frac{2}{\mu}(c^2 - a_0 + \frac{\alpha\beta}{\gamma})$ and the sign of μ we can summarize the bounds for the extrema in the following form:

(i) $\min(g_1, g_2) < w_{min} < w_{max} < \max(g_1, g_2)$ for $0 < c^2 \le a_0 - \frac{2\alpha\beta}{\gamma}$,

(ii) $\min(g_3, g_4) < w_{min} < w_{max} < \max(g_3, g_4)$ for $a_0 - \frac{2\alpha\beta}{\gamma} < c^2 < a_0 - \frac{\alpha\beta}{\gamma}$,

(iii) $\min(g_3, g_4) < w_{min} < w_{max} < \max(g_3, g_4)$ for $a_0 - \frac{\alpha\beta}{\gamma} < c^2 \le a_0 - \frac{2\alpha\beta}{3\gamma}$,

(iv) $\min(g_1, g_5) < w_{min} < w_{max} < \max(g_1, g_5)$ for $a_0 - \frac{2\alpha\beta}{3\gamma} < c^2 < a_0$,

(v) $\min(g_1, g_5) < w_{min} < w_{max} < \max(g_1, g_5)$ for $a_0 < c^2$,

where $g_1, \ldots, g_5$ are defined as follows:

$$g_1 = \frac{c^2 - a_0}{\mu}, \quad g_2 = \frac{c^2 - a_0}{\mu} + \frac{8\alpha\beta + \sqrt{72\gamma^2(c^2 - a_0 + \frac{4\alpha\beta}{3\gamma})(c^2 - a_0 + \frac{2\alpha\beta}{3\gamma})}}{6\gamma\mu},$$
$$g_3 = 2\left(\frac{c^2 - a_0}{\mu} + \frac{\alpha\beta}{\mu\gamma}\right), \quad g_4 = \frac{2(\sqrt{\alpha\beta(3\gamma(a_0 - c^2) - 2\alpha\beta)} - \alpha\beta)}{3\gamma\mu},$$
$$g_5 = \frac{c^2 - a_0}{\mu} + \frac{8\alpha\beta - \sqrt{72\gamma^2(c^2 - a_0 + \frac{4\alpha\beta}{3\gamma})(c^2 - a_0 + \frac{2\alpha\beta}{3\gamma})}}{6\gamma\mu}. \tag{3.16}$$

2. Waves related to the equilibrium point $(w; z) = (\frac{2}{\mu}(c^2 - a_0 + \frac{\alpha\beta}{\gamma}); 0)$.

By means of the linearisation technique the following conditions for the velocity are obtained:

1. for normal dispersion, i.e., $\gamma a_0 - \gamma a_1 - \alpha\beta > 0$ and

$$\text{either } a)\ c^2 \ge a_0 > a_0 - \frac{\alpha\beta}{\gamma} > a_1 \quad \text{or } b)\ a_0 > c^2 > a_0 - \frac{\alpha\beta}{\gamma} > a_1,$$
$$\text{or } c)\ a_0 - \frac{\alpha\beta}{\gamma} > a_1 > c^2 > a_0 - \frac{2\alpha\beta}{\gamma} \ge 0, \quad \text{or } d)\ a_0 - \frac{2\alpha\beta}{\gamma} > c^2 > a_1; \tag{3.17}$$

2. for dispersionless case, i.e., $\gamma a_0 - \gamma a_1 - \alpha\beta = 0$,

$$c^2 > a_0 - 2\alpha\beta/\gamma; \tag{3.18}$$

3. for anomalous dispersion, i.e., $\gamma a_0 - \gamma a_1 - \alpha\beta < 0$ and

$$\text{either a) } c^2 = a_0 > a_1 > a_0 - \alpha\beta/\gamma, \quad \text{or b) } c^2 > a_0, a_1 > a_0 - \alpha\beta/\gamma,$$
$$\text{or c) } a_0 > c^2 > a_1 > a_0 - \alpha\beta/\gamma,$$
$$\text{or d) } a_0, a_1 > a_0 - \alpha\beta/\gamma > c^2 > a_0 - 2\alpha\beta/\gamma \ge 0. \tag{3.19}$$

Taking the singular value of $y_s = \frac{\alpha(c^2-a_1)}{\delta^{1/2}\lambda}$ of y into account, we again deduce restrictions for the extrema of w. To this end, let us set in (3.11) $w_2 = w_{ext}$, $w_1 = \frac{2}{\mu}(c^2 - a_0 + \frac{\alpha\beta}{\gamma})$ and $y_1 = y(\frac{2}{\mu}(c^2 - a_0 + \frac{\alpha\beta}{\gamma}))$. Then we obtain

$$\left(\frac{\delta(c^2-a_1)}{2} - \frac{\delta^{3/2}\lambda}{3\alpha}y_1\right)y_1^2 = R_1 w_{ext}^4 + R_2 w_{ext}^3 + R_3 w_{ext}^2 + R_4, \tag{3.20}$$

where R_1, R_2, R_3 are defined by (3.12) and

$$R_4 = \frac{2\alpha\beta(\gamma(c^2-a_0)+\alpha\beta)^3}{3\gamma^3\mu^2}.$$

From this relation we derive the following implicit estimate:

$$\begin{aligned}\lambda^2 &< \frac{R_0}{R_1 w_{max}^4 + R_2 w_{max}^3 + R_3 w_{max}^2 + R_4} \\ &= \frac{R_0}{R_1 w_{min}^4 + R_2 w_{min}^3 + R_3 w_{min}^2 + R_4},\end{aligned} \tag{3.21}$$

where R_0 is given by (3.14).

Furthermore, similarly as in the previous case, we obtain the following bounds:

(vi) $\min(g_1, g_5) < w_{min} < w_{max} < \max(g_1, g_5)$ for $0 < c^2 < a_0 - \frac{2\alpha\beta}{\gamma}$,

(vii) $\min(g_1, g_5) < w_{min} < w_{max} < \max(g_1, g_5)$ for $a_0 - \frac{2\alpha\beta}{\gamma} < c^2 < a_0 - \frac{4\alpha\beta}{3\gamma}$,

(vii) $\min(g_6, g_7) < w_{min} < w_{max} < \max(g_6, g_7)$ for $a_0 - \frac{4\alpha\beta}{3\gamma} \le c^2 < a_0 - \frac{\alpha\beta}{\gamma}$,

(viii) $\min(g_6, g_7) < w_{min} < w_{max} < \max(g_6, g_7)$ for $a_0 - \frac{\alpha\beta}{\gamma} < c^2 \le a_0$,

(ix) $\min(g_1, g_2) < w_{min} < w_{max} < \max(g_1, g_2)$ for $a_0 < c^2$,

where g_1, g_2, g_5 are defined by (3.16), $g_6 = 0$ and

$$g_7 = \frac{2(\sqrt{4\alpha^2\beta^2 + 3\alpha\beta\gamma(c^2-a_0)} + 2\alpha\beta + 3\gamma(c^2-a_0))}{3\gamma\mu}.$$

To conclude this subsection, we point out that the conditions (3.8)–(3.10) (resp. (3.17)–(3.19)) give ranges of the velocity when periodic wave may exist. The restrictions for the extrema (i.e. the minima and maxima) of the wave depend on the nonlinearity parameters μ and λ. In addition to the condition (3.13) (resp. (3.21)), the maxima and minima must satisfy the inequalities (i)–(v) (resp. (vi)–(ix)). The bigger λ and μ, the smaller the range of the extrema. In case $\lambda = 0$ the inequality (3.13) (resp. (3.21)) drops.

4 Solitary and Related Waves

4.1 Solitary waves

As it was mentioned the system (2.5) is supposed to have a solitary wave solution, i.e. a solution that satisfies $w \to 0$ as $|\eta| \to \infty$. A solitary wave solution can exist only if parameter μ does not equal zero. This condition can be easily verified by integrating the equation (2.9) with the lower and upper limits $y(w_{max}) = y(w_{min}) = 0$.

In the mathematical sense, the solitary wave is a limit case of the periodic wave related to the equilibrium point $w = (\frac{2}{\mu}(c^2 - a_0 + \frac{\alpha\beta}{\gamma}); 0)$ whose phase curve approaches the origin $(0; 0)$. Let us consider the relation (3.20) deduced for such a periodic wave. By subtraction we immediately get (3.15). Taking either the limit $w_{min} \to 0$ or $w_{max} \to 0$ there, we reach the following quartic equation

$$w_{amp}^4 - \frac{3\mu\gamma(c^2 - a_0) + 2\mu\alpha\beta}{6} w_{amp}^3 + \frac{(c^2 - a_0)(\gamma(c^2 - a_0) + \alpha\beta)}{2} w_{amp}^2 = 0,$$

where $w_{amp} = w_{max}$ in the case of the positive wave and $w_{amp} = w_{min}$ in the case of the negative wave. The nontrivial roots of this equation give the possible values for the amplitude:

$$w_{amp12} = \frac{2(\pm\sqrt{4\alpha^2\beta^2 + 3\alpha\beta\gamma(c^2 - a_0)} + 2\alpha\beta + 3\gamma(c^2 - a_0))}{3\gamma\mu}. \tag{4.1}$$

In order w_{amp12} to be real, the velocity must satisfy the inequality $c^2 \geq a_0 - \frac{4\alpha\beta}{3\gamma}$. A detailed analysis of the behavior of the system (3.1) near the equilibrium point $(w; z) = \left(\frac{2}{\mu}(c^2 - a_0 + \frac{\alpha\beta}{\gamma}); 0\right)$ enables to extract the right formula for amplitude of the wave:

$$w_{amp} = \frac{2(\sqrt{4\alpha^2\beta^2 + 3\alpha\beta\gamma(c^2 - a_0)} + 2\alpha\beta + 3\gamma(c^2 - a_0))}{3\gamma\mu}. \tag{4.2}$$

Inserting this formula to (3.21), a more simple condition for λ follows:

$$|\lambda| < \sqrt{\frac{\alpha^2\mu^2\gamma^3(c^2 - a_1)^3}{4\alpha\beta(\alpha\beta + \gamma(c^2 - a_0))^3}}. \tag{4.3}$$

To deduce conditions for the velocity c, we begin with the consideration that the singular value $\frac{c^2 - a_0}{\mu}$ of w cannot be located between 0 and the equilibrium value $\frac{2}{\mu}(c^2 - a_0 + \frac{\alpha\beta}{\gamma})$. This implies $c^2 < a_0$. Furthermore, since the phase curve of the solitary wave approaches the origin point $(0, 0)$ as $|\eta| \to \infty$, we can again linearize the system (3.1) at this point. The vanishing solution occurs only in case $\frac{\gamma(c^2 - a_0) + \alpha\beta}{(c^2 - a_1)(c^2 - a_0)} < 0$. Combining this condition with the inequalities $c^2 < a_0$ and $c^2 \geq a_0 - \frac{4\alpha\beta}{3\gamma}$, obtained before, we derive the following restrictions for the velocity:

1. for normal dispersion, i.e., $\gamma a_0 - \gamma a_1 - \alpha\beta > 0$,

$$\begin{aligned} &\text{either a) } a_0 > c^2 > a_0 - \alpha\beta/\gamma > a_1 \\ &\text{or} \quad \text{b) } a_0 - \alpha\beta/\gamma > a_1 > c^2 \geq a_0 - 4\alpha\beta/(3\gamma); \end{aligned} \tag{4.4}$$

2. for dispersionless case, i.e., $\gamma a_0 - \gamma a_1 - \alpha\beta = 0$,

$$a_0 > c^2 \geq a_0 - 4\alpha\beta(3\gamma); \tag{4.5}$$

3. for anomalous dispersion, i.e., $\gamma a_0 - \gamma a_1 - \alpha\beta < 0$,

$$\begin{aligned} &\text{either a) } a_0 > c^2 > a_1 > a_0 - \alpha\beta/\gamma \\ &\text{or} \quad \text{b) } a_1 > a_0 - \alpha\beta/\gamma > c^2 \geq a_0 - 4\alpha\beta/(3\gamma). \end{aligned} \tag{4.6}$$

We mention that the same conditions for the velocity and the parameter λ were deduced also in [12], but by means of different techniques.

Moreover, we underline that the numerical solitary wave solution is unstable near the point $(w; z) = (0; 0)$, so instead of the pure numerical solution in practice the following (piecewise analytical-numerical) approximation is used:

$$w(\eta) = \begin{cases} w_{-\infty}(\eta) = w(-\hat{\eta}_1)e^{\sqrt{\kappa}(\eta+\hat{\eta}_1)} & \eta \leq -\hat{\eta}_1, \\ w(\eta) & -\hat{\eta}_1 \leq \eta \leq \hat{\eta}_2, \\ w_{+\infty}(\eta) = w(\hat{\eta}_2)e^{-\sqrt{\kappa}(\eta-\hat{\eta}_2)} & \hat{\eta}_2 \leq \eta, \end{cases} \tag{4.7}$$

where $\kappa = -\frac{\gamma(c^2-a_0)+\alpha\beta}{\delta(c^2-a_1)(c^2-a_0)}$ and $\hat{\eta}_j$, $j = 1, 2$, are sufficiently large numbers.

4.2 Another wave of infinite length

An interesting wave of infinite length can be obtained from the periodic wave related to the equilibrium point $(0; 0)$ in case the phase curve approaches $w = (\frac{2}{\mu}(c^2 - a_0 + \frac{\alpha\beta}{\gamma}); 0)$. Such a wave approaches a nonzero constant as $|\eta| \to \infty$, hence it is not a solitary wave in the classical sense. Physically, this wave may occur in predeformed materials.

Since the phase curve turns around the origin, the wave changes the sign around $\eta = 0$. More precisely, the following inequalities hold for those wave:

$$\frac{\gamma(c^2 - a_0) + \alpha\beta}{\delta(c^2 - a_1)(c^2 - a_0)} > 0, \qquad \frac{\gamma(\gamma(c^2 - a_0) + \alpha\beta)}{\delta(c^2 - a_1)(\gamma(c^2 - a_0) + 2\alpha\beta)} < 0$$

and one of the following conditions must be satisfied:

$$\begin{aligned} &\text{either} \quad \mu > 0, \quad \frac{c^2 - a_0}{\mu} < 0, \quad \frac{c^2 - a_0}{\mu} < 2\left(\frac{c^2 - a_0}{\mu} + \frac{\alpha\beta}{\gamma\mu}\right) \\ &\text{or} \quad \mu < 0, \quad \frac{c^2 - a_0}{\mu} > 0, \quad \frac{c^2 - a_0}{\mu} > 2\left(\frac{c^2 - a_0}{\mu} + \frac{\alpha\beta}{\gamma\mu}\right). \end{aligned}$$

The solution obeys the following properties:

$$w(0) = \frac{2\big(\sqrt{\alpha\beta(3\gamma(a_0 - c^2) - 2\alpha\beta)} - \alpha\beta\big)}{3\gamma\mu}, \quad w'(0) = 0,$$

$$\lim_{\eta\to\pm\infty} w(\eta) = 2\left(\frac{c^2 - a_0}{\mu} + \frac{\alpha\beta}{\gamma\mu}\right), \quad \lim_{\eta\to\pm\infty} w^{(i)}(\eta) = 0. \tag{4.8}$$

Restrictions for the velocity are

$$\begin{aligned} &\text{either } c^2 > a_1, \quad a_0 - 2\alpha\beta/\gamma < c^2 < a_0 - \alpha\beta/\gamma \\ &\text{or} \quad\; c^2 < a_1, \quad a_0 - \alpha\beta/\gamma < c^2 < a_0 - 2\alpha\beta/(3\gamma) \end{aligned} \tag{4.9}$$

and the condition for λ is

$$|\lambda| < \sqrt{\frac{\alpha^2\mu^2\gamma^3(a_1 - c^2)^3}{4\alpha\beta(\alpha\beta + \gamma(c^2 - a_0))^3}}. \tag{4.10}$$

The stable approximation of this solution is

$$w(\eta) = \begin{cases} w_{-\infty}(\eta) = w(-\hat\eta_1)e^{\sqrt{\kappa}(\eta+\hat\eta_1)} + \Theta\big(1 - e^{\sqrt{\kappa}(\eta+\hat\eta_1)}\big) & \eta \le -\hat\eta_1, \\ w(\eta) & -\hat\eta_1 \le \eta \le \hat\eta_2, \\ w_{+\infty}(\eta) = w(\hat\eta_2)e^{-\sqrt{\kappa}(\eta-\hat\eta_2)} + \Theta\big(1 - e^{-\sqrt{\kappa}(\eta-\hat\eta_2)}\big) & \hat\eta_2 \le \eta. \end{cases}$$

Here

$$\kappa = -\frac{\gamma(\gamma(c^2 - a_0) + \alpha\beta)}{\delta(c^2 - a_1)(\gamma(c^2 - a_0) + 2\alpha\beta)}, \quad \Theta = 2\Big(\frac{c^2 - a_0}{\mu} + \frac{\alpha\beta}{\gamma\mu}\Big), \quad \hat\eta_j,\ j = 1, 2,$$

are again sufficiently large numbers.

5 Inverse Problems

5.1 Description of method

Solving an inverse problem is understood as finding (extracting) values of parameters of an equation or a system of them (i.e. in this article finding the value of a_0, a_1, γ, α, β, μ and λ on condition that the parameters c and δ are supposed to be known with high accuracy and to be controlled) from the experimental (simulation) data. The number of parameters and their accuracy depend on how the data is presented. We use the periodic or solitary waves and assume that the data can contain information either about the macrodeformation only, or about the micro- and macrodeformations. Also the data can be given either as wave profiles or as a collection of characteristics, i.e. the amplitudes of waves, the lengths etc.

From the practical viewpoint, it is important that the traveling waves that we use, are orbitally stable, because otherwise these waves are very difficult to observe. Rigorous mathematical proof of the stability is a complicated task, because the system (2.1) is not integrable. The stability can be seen from numerical simulations [17, 21, 22]. Moreover, physical solitary waves are observed in some microstructured materials [18, 19].

As it was mentioned, analytical solutions of the problems (2.5) and (2.8) are not realistic to find. Therefore, it remains to solve them numerically. To find the parameters a_0, a_1, γ, α, β, μ and λ we have to fit the computed solution to the experimental data, i.e. we have to use methods analogous to the shooting method and methods of minimization. We must underline that there are some tricks which allow us to extract a part of parameters analytically, but their accuracy is too low, so we will not consider them.

As for the method used to find the parameter, we have to construct the special positive objective functional (it is special because of the dependence on the presentation of data and the number of parameters and waves) and numerically minimize it. The minimal value of functional is either zero in case of the data are not affected by the noise or the smallest (as possible) positive number in the case of presence of the noise. According to that we obtain either an exact or a quasi-solution.

Let us start with the case when the data contain information exclusively about the wave profile of the macrodeformation. Then the basic system (2.8) contains the three parameters α, β and λ only in the form of the quotient $\frac{\lambda}{\alpha}$ and the product $\alpha\beta$. Therefore, the values of these three parameters cannot be extracted from macro-measurements. The vector of unknown physical parameters contains 6 components: a_0, a_1, $\alpha\beta$, γ, $\frac{\lambda}{\alpha}$, μ. To solve the inverse problem, we have to minimize the following functional:

$$F(P) = \sqrt{\sum_{i,s} \left(\hat{w}(\eta_i, c_s) - w(\eta_i, c_s, P)\right)^2} + \sum_{k,s} G_k(P, c_s), \tag{5.1}$$

where $P^T = (p_1, \ldots, p_n)$ is a vector of n parameters to be determined (below we see that this may contain Cauchy data in addition to physical parameters), c_s is the velocity of the s-th wave (the number of waves in this case must be greater or equal to three), G_k are penalty functions which are zero if the physical constraints (2.2) and (2.3) are fulfilled and positive and grow very fast if not, $w(\eta_i, c_s, P)$ is the computed wave solution at $\eta = \eta_i$ corresponding to given P, c_s and $\hat{w}(\eta_i, c_s)$ is the measured data (in our case generated by simulation).

The simulated data are as follows: $\hat{w}(\eta_i, c_s) = w(\eta_i, c_s, P^\dagger) + \epsilon_i$ where ϵ_i is the noise and $P^\dagger$ is the prefixed "exact solution" of the inverse problem.

As we mentioned, the Cauchy data for the system (3.1) may also be unknown. This means that, in general, the vector P consists of the following components:

$$\begin{aligned} &p_1 = a_0, \quad p_2 = a_1, \quad p_3 = \alpha\beta, \quad p_4 = \gamma, \quad p_5 = \lambda/\alpha, \quad p_6 = \mu, \\ &p_{5+2s} = w_s(\eta_0), \quad p_{6+2s} = w_s'(\eta_0), \quad s = 1, \ldots, J, \end{aligned} \tag{5.2}$$

where J is the number of waves incorporated. In special cases the number of unknowns can be reduced (if either μ or λ is known to be zero or both of them are known to be zero or Cauchy conditions are either same for all waves or exactly known).

Secondly, let us consider the case when the wave profiles of both macro- and microdeformation are given. Then the whole vector of physical data a_0, a_1, α, β, γ, λ, μ can be recovered, because the additionally given microdeformation depends on α separately from the product $\alpha\beta$ (see formula (2.7)). Now the objective functional reads

$$F(P) = \sqrt{\sum_{i,s}\big(\hat{w}(\eta_i, c_s) - w(\eta_i, c_s, P)\big)^2 + \sum_{i,s}\big(\hat{\varphi}(\eta_i, c_s) - \varphi(\eta_i, c_s, P)\big)^2} \\ + \sum_{k,s} G_k(P, c_s),$$

where P again is a vector of parameters to be determined, c_s is the velocity, G_k are the penalty functions, $w(\eta_i, c_s, P)$, $\varphi(\eta_i, c_s, P)$ are the macro- and micro-components of the computed wave and $\hat{w}(\eta_i, c_s)$, $\hat{\varphi}(\eta_i, c_s)$ are the measured data. The simulated data are constructed in a manner similar to the previous case.

Finally, if the data contain a collection of characteristics of waves then the functional analogical to (5.1) may have the form

$$F(P) = \sqrt{\sum_{i,s}\big(\hat{K}_i(c_s) - K_i(c_s, P)\big)^2} + \sum_{k,s} G_k(P, c_s). \tag{5.3}$$

Here K_1 and K_2 are the minimum and the maximum of the periodic wave of a macrocomponent (in case of both micro- and macrodeformation there are also K_5 and K_6, i.e. the minimum and the maximum of the periodic wave of a microcomponent) and K_3 and K_4 are the half-lengths, i.e. $w(\eta' - K_3) = w(\eta' + K_4) = K_1$ (or $\varphi(\eta' - K_3) = \varphi(\eta' + K_4) = K_5$) where η' is some value of η where the wave-function attains the maximum K_2. Note that now the number of parameters is $n = 6 + J$ (in this case for simplicity we assume that $p_{6+2s} = 0$) and can be reduced on some conditions.

5.2 Numerical results

In case of wave profiles the number of points of a wave profile is denoted by N. Also we define the noise as $\epsilon_i = kR \min_s |w(c_s, P^\dagger)_{max} - w(c_s, P^\dagger)_{min}|$, where R is the uniformly distributed pseudorandom number in the interval $[-1, 1]$ and the coefficient k is specified for every case.

In case of a collection of wave characteristics we define two types of noise: $\epsilon_A = k_1 R \times \min_s |w(c_s, P^\dagger)_{max} - w(c_s, P^\dagger)_{min}|$ for wave amplitudes and $\epsilon_L = k_2 R \min_s |T(c_s, P^\dagger)|$ for wave lengths, where k_1 and k_2 are specified for every case and $T(c_s, P^\dagger)$ is the length of the s-th wave.

Evidently, a single wave doesn't contain enough information to recover all unknown parameters whatever is the number of measured points or characteristics of this wave. Indeed, the basic equation of the macro-component of the traveling wave (2.9) has 5 degrees of freedom (coefficients of yy', y^2y', w, w^2

Table 1. Maximal absolute errors (wave profile, macrodeformation only).

$N = 150$, $T = 14.8491$, $k = 0.005$, $\max\lvert\epsilon\rvert = 0.00826$, $\delta = 1$						
Number of waves		3	4	5	6	7
Exact values		Maximal absolute errors				
a_0	3	0.03151	0.01648	0.01763	0.01781	0.01769
a_1	1	0.09697	0.05569	0.04943	0.02927	0.04370
$\alpha\beta$	1	0.05932	0.03517	0.03259	0.02390	0.02681
γ	1	0.00663	0.00487	0.00454	0.00280	0.00269
μ	1	0.01747	0.00888	0.00910	0.00915	0.00808
λ/α	1	0.02497	0.02220	0.02012	0.01665	0.01099
w_i	1	0.00103	0.00076	0.00065	0.00089	0.00068
w_i'	0	0.00130	0.00130	0.00124	0.00075	0.00078

and w^3). But the number of unknown parameters contained in this equation is 6 (i.e. a_0, a_1, $\alpha\beta$, γ, $\frac{\lambda}{\alpha}$ and μ). Therefore, at least 2 waves with different velocities have to be measured. Similar situation occurs if both macro- and micro-components of the waves are incorporated.

Increasing the number of measured waves reduces the error of the solution of the inverse problem. The reasons are that that then the amount of information used in the problem is bigger and the influence of stochastic errors of measurements is smaller (the mathematical expectations of the errors of the measurements are equal to 0).

The 50 simulations are done for every case. The minimization of related objective functionals was performed using the Nelder–Mead algorithm [1]. Number T in the tables denotes the maximal wave-length, i.e. $T = \max_s T(c_s, P^\dagger)$.

In the examples the exact parameters and the velocities are chosen so that the theoretical existence conditions for the periodic waves are satisfied. More precisely, the range of velocity of the measured waves is the interval $[2.2, 2.8]$, where the particular velocities are taken by the formula $c_j = 2.2 + \tau(j-1)$, $j = 1, \ldots, M$, where $\tau = \frac{2.8-2.2}{M-1}$ and M is the number of measured waves.

Tables 1 and 2 show absolute errors of numerical results that are obtained solving inverse problems that use measurements of N points on the profiles of different number of measured waves (from 3 to 7). Tables contain maximal absolute errors of the parameters. In addition to the physical parameters, Cauchy data of all waves are assumed to be unknown (the parameters w_i and w_i').

Tables 3 and 4 contain absolute errors of results obtained by means of measurements of wave characteristics (minima, maxima and half-lengths). Since the results are worse than in the previous case, the number of measured waves is increased (from 5 to 30).

The results show that the informativity of measured waves is very different with respect to different parameters. Wave profiles are much more informative than the wave characteristics. The biggest error has the parameter a_1. In case only the macrodeformation is measured, the best results are obtained for γ. In case both macro- and microdeformation are measured, the results are better. In particular, then the parameters α and β are separated form the product and

Table 2. Maximal absolute errors (wave profile, macro- and microdeformation).

$N = 150$, $T = 14.8491$, $k = 0.005$, $\max\|\epsilon\| = 0.00826$, $\delta = 1$						
Number of waves		3	4	5	6	7
Exact values		Maximal absolute errors				
a_0	3	0.00514	0.00609	0.00384	0.00311	0.00405
a_1	1	0.04250	0.01630	0.04345	0.03570	0.02361
α	1	0.00079	0.00105	0.00087	0.00084	0.00153
β	1	0.03010	0.01032	0.02625	0.02168	0.01400
γ	1	0.00175	0.00114	0.00270	0.00239	0.00160
μ	1	0.01150	0.01198	0.00694	0.00500	0.00553
λ	1	0.00499	0.00324	0.00598	0.00561	0.00389
w_i	1	0.00131	0.00143	0.00087	0.00089	0.00107
w_i'	0	0.00030	0.00038	0.00034	0.00042	0.00047

Table 3. Maximal absolute errors (wave characteristics, macrodeformation only).

$\max\|\epsilon_A\| = 0.00165$, $\max\|\epsilon_L\| = 0.00975$, $k_1 = k_2 = 0.001$, $\delta = 1$					
Number of waves		5	10	20	30
Exact values		Maximal absolute errors			
a_0	3	0.07604	0.05938	0.04131	0.01876
a_1	1	0.56430	0.25994	0.24591	0.19478
$\alpha\beta$	1	0.19849	0.19849	0.18713	0.13654
γ	1	0.01390	0.00804	0.00906	0.00661
μ	1	0.05486	0.02625	0.02354	0.01452
λ/α	1	0.07038	0.03470	0.03587	0.03434

the best results are obtained for α. We point out that the numerical results are comparable with the results obtained formerly for inverse problems for solitary waves [12].

6 Conclusions

The microstructure has a dispersive impact to the wave propagation. Under a proper balance between the dispersion and the nonlinearity, periodic and solitary waves may occur. In this paper we deduced existence conditions for such waves. Here the crucial role play the nonlinearity parameters μ and λ, corresponding to the macro- and micro-levels, respectively. In case $\mu = 0$ only a single family of periodic waves exist and solitary waves do not occur. In case $\mu \neq 0$ two families of periodic waves exist and the extrema of the periodic waves have bounds that depend on μ (see conditions (i)–(ix)). Approaching these bounds the periodic wave attains infinite length and becomes either a solitary wave or a wave described in Subsection 5.2.

The micro-level parameter λ rather disturbs the balance between the nonlinearity and the dispersion. The relations (3.13), (3.21), (4.3) and (4.10) show

Table 4. Maximal absolute errors (wave characteristics, macro- and microdeformation).

$\max\|\epsilon_A\| = 0.00165$, $\max\|\epsilon_L\| = 0.00975$, $k_1 = k_2 = 0.001$, $\delta = 1$					
Number of waves		5	10	20	30
Exact values		Maximal absolute errors			
a_0	3	0.00581	0.00382	0.00356	0.00150
a_1	1	0.13040	0.10708	0.05625	0.03569
α	1	0.00140	0.00098	0.00100	0.00044
β	1	0.07860	0.06552	0.03979	0.02140
γ	1	0.00933	0.00516	0.00383	0.00304
μ	1	0.00568	0.00310	0.00452	0.00232
λ	1	0.02346	0.01797	0.01102	0.00568

that $|\lambda|$ must be sufficiently small. Increasing $|\lambda|$ to a certain critical level, the balance breaks.

Moreover, we showed that periodic and solitary waves can be used to reconstruct the physical parameters of the material. The numerical tests insist that the measurements of the wave profiles contain enough information to recover the parameters with acceptable accuracy. The measurements of wave characteristics (extrema and half-lengths) are less informative. Some parameters can be well-recovered, but some other parameters are very sensitive with respect to measurement errors.

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