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Positive Solutions for Singular Systems of Higher-Order Multi-Point Boundary Value Problems

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Abstract. We investigate the existence of positive solutions for systems of singular nonlinear higher-order differential equations subject to multi-point boundary conditions.

Keywords: higher-order differential system, singular, multi-point boundary conditions, positive solutions.

AMS Subject Classification: 34B10; 34B18.

1 Introduction

We consider the singular system of nonlinear higher-order ordinary differential equations

$$\begin{cases} u^{(n)}(t) + f\big(t, v(t)\big) = 0, & t \in (0,T),\ n \in \mathbb{N},\ n \geq 2, \\ v^{(m)}(t) + g\big(t, u(t)\big) = 0, & t \in (0,T),\ m \in \mathbb{N},\ m \geq 2, \end{cases} \tag{S}$$

with the multi-point boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(T) = \displaystyle\sum_{i=1}^{p-2} a_i u(\xi_i), \quad p \in \mathbb{N},\ p \geq 3, \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(T) = \displaystyle\sum_{i=1}^{q-2} b_i v(\eta_i), \quad q \in \mathbb{N},\ q \geq 3. \end{cases} \tag{BC}$$

We present some weaker assumptions on f and g, which do not possess any sublinear or superlinear growth conditions and may be singular at $t = 0$

and/or $t = T$, such that positive solutions for problem (S)–(BC) exist. By a positive solution of (S)–(BC), we understand a pair of functions $(u, v) \in (C([0,T];\mathbb{R}_+) \cap C^n((0,T))) \times (C([0,T];\mathbb{R}_+) \cap C^m((0,T)))$ satisfying (S) and (BC) with

$$\sup_{t\in[0,T]} u(t) > 0, \qquad \sup_{t\in[0,T]} v(t) > 0.$$

This problem is a generalization of the problem studied in [7], where $n = m = 2$. In [12], the authors investigated the existence of positive solutions for system (S) with $n = m = 2$ and the boundary conditions

$$u(0) = 0,\ u(1) = \alpha u(\eta),\ v(0) = 0,\ v(1) = \alpha v(\eta),\quad \eta \in (0,1), 0 < \alpha\eta < 1.$$

In [16], the authors studied the existence and multiplicity of positive solutions for system (S) with $n = m = 2$, $T = 1$ and boundary conditions which contain only one intermediate point. We also mention the paper [14], where the authors used the fixed point index theory to prove the existence of positive solutions for the system (S) with $f(t, v(t))$ and $g(t, u(t))$ replaced by $c(t)\widetilde{f}(u(t), v(t))$ and $d(t)\widetilde{g}(u(t), v(t))$, respectively, and (BC), where $\frac{1}{2} \le \xi_1 < \xi_2 < \cdots < \xi_{p-2} < 1$, $\frac{1}{2} \le \eta_1 < \eta_2 < \cdots < \eta_{q-2} < 1$ $(T = 1)$. Other systems with various nonlocal boundary conditions were investigated in the papers [2, 3, 5, 8, 9, 15]. Some multi-point boundary value problems for systems of ordinary differential equations which involve positive eigenvalues were studied in recent years by using the Guo–Krasnosel'skii fixed point theorem. In [4], the authors give sufficient conditions for λ, μ, f and g such that the system

$$\begin{cases} u^{(n)}(t) + \lambda c(t) f\big(u(t), v(t)\big) = 0, & t \in (0,T),\ n \in \mathbb{N},\ n \ge 2, \\ v^{(m)}(t) + \mu d(t) g\big(u(t), v(t)\big) = 0, & t \in (0,T),\ m \in \mathbb{N},\ m \ge 2, \end{cases} \tag{S_1}$$

with the boundary conditions (BC) has positive solutions ($u(t) \ge 0$, $v(t) \ge 0$ for all $t \in [0,T]$ and $(u,v) \ne (0,0)$). The system (S_1) with $n = m = 2$ and the multi-point boundary conditions

$$\begin{cases} \alpha u(0) - \beta u'(0) = 0, \quad u(T) = \displaystyle\sum_{i=1}^{p-2} a_i u(\xi_i), \quad p \in \mathbb{N},\ p \ge 3, \\ \gamma v(0) - \delta v'(0) = 0, \quad v(T) = \displaystyle\sum_{i=1}^{q-2} b_i v(\eta_i), \quad q \in \mathbb{N},\ q \ge 3, \end{cases} \tag{BC_1}$$

has been investigated in [6].

In recent years, multi-point boundary value problems for second-order or higher-order differential or difference equations/systems have been investigated by many authors, by using different methods such as fixed point theorems in cones, the Leray–Schauder continuation theorem and its nonlinear alternatives, and the coincidence degree theory.

In Section 2, we shall present some auxiliary results which investigate two boundary value problems for higher-order equations (the problems (2.1)–(2.2) and (2.4)–(2.5) below). In Section 3, we shall prove two existence results for the positive solutions with respect to a cone for our problem (S)–(BC), which are based on the Guo–Krasnosel'skii fixed point theorem, presented below.

Theorem 1. *Let X be a Banach space and let $C \subset X$ be a cone in X. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and let $\mathcal{A} : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ be a completely continuous operator such that, either*

i) $\|\mathcal{A}u\| \le \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \ge \|u\|$, $u \in C \cap \partial\Omega_2$, *or*

ii) $\|\mathcal{A}u\| \ge \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \le \|u\|$, $u \in C \cap \partial\Omega_2$.

Then $\mathcal{A}$ has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Finally, in Section 4, we shall present some examples which illustrate our main results.

2 Auxiliary Results

In this section, we shall present some auxiliary results from [10,11] (see also [13]) related to the following nth-order differential equation with p-point boundary conditions

$$u^{(n)}(t) + y(t) = 0, \quad t \in (0,T), \tag{2.1}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(T) = \sum_{i=1}^{p-2} a_i u(\xi_i). \tag{2.2}$$

Lemma 1. *If $d = T^{n-1} - \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \ne 0$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$ and $y \in C([0,T])$, then the solution of (2.1)–(2.2) is given by*

$$u(t) = \frac{t^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} y(s)\,ds - \frac{t^{n-1}}{d(n-1)!} \sum_{i=1}^{p-2} a_i \int_0^{\xi_i} (\xi_i - s)^{n-1} y(s)\,ds$$
$$- \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y(s)\,ds, \quad 0 \le t \le T.$$

Lemma 2. *Under the assumptions of Lemma 1, the Green's function for the boundary value problem (2.1)–(2.2) is given by*

$$G_1(t,s) = \begin{cases} \frac{t^{n-1}}{d(n-1)!}\left[(T-s)^{n-1} - \sum_{i=j+1}^{p-2} a_i (\xi_i - s)^{n-1}\right] - \frac{1}{(n-1)!}(t-s)^{n-1}, \\ \quad \text{if } \xi_j \le s < \xi_{j+1},\ s \le t, \\ \frac{t^{n-1}}{d(n-1)!}\left[(T-s)^{n-1} - \sum_{i=j+1}^{p-2} a_i (\xi_i - s)^{n-1}\right], \\ \quad \text{if } \xi_j \le s < \xi_{j+1},\ s \ge t,\ j = 0, \ldots, p-3, \\ \frac{t^{n-1}}{d(n-1)!}(T-s)^{n-1} - \frac{1}{(n-1)!}(t-s)^{n-1}, \quad \text{if } \xi_{p-2} \le s \le T,\ s \le t, \\ \frac{t^{n-1}}{d(n-1)!}(T-s)^{n-1}, \quad \text{if } \xi_{p-2} \le s \le T,\ s \ge t\ (\xi_0 = 0), \end{cases}$$

for all $(t,s) \in [0,T] \times [0,T]$.

Using the Heaviside function on $\mathbb{R}$, $H(x) = 1$ for $x \geq 0$, and $H(x) = 0$ for $x < 0$, the above Green's function can be written in a compact form

$$G_1(t,s) = \frac{t^{n-1}}{d(n-1)!}\left[(T-s)^{n-1} - \sum_{i=1}^{p-2} a_i(\xi_i - s)^{n-1}H(\xi_i - s)\right] - \frac{1}{(n-1)!}(t-s)^{n-1}H(t-s), \quad (t,s) \in [0,T]\times[0,T].$$

By using the above Green's function the solution of problem (2.1)–(2.2) is expressed as $u(t) = \int_0^T G_1(t,s)y(s)\,ds$.

Lemma 3. *Under the assumptions of Lemma* 1*, the Green's function for the boundary value problem* (2.1)*–*(2.2) *can be expressed as*

$$G_1(t,s) = g_1(t,s) + \frac{t^{n-1}}{d}\sum_{i=1}^{p-2} a_i g_1(\xi_i, s),$$

where

$$g_1(t,s) = \frac{1}{(n-1)!T^{n-1}}\begin{cases} t^{n-1}(T-s)^{n-1} - T^{n-1}(t-s)^{n-1}, & 0 \leq s \leq t \leq T, \\ t^{n-1}(T-s)^{n-1}, & 0 \leq t \leq s \leq T. \end{cases} \tag{2.3}$$

Lemma 4. *The function g_1 given in* (2.3) *has the properties*:

a) $g_1 : [0,T]\times[0,T] \to \mathbb{R}_+$ *is a continuous function and* $g_1(t,s) \geq 0$ *for all* $(t,s) \in [0,T]\times[0,T]$.

b) $g_1(t,s) \leq g_1(\theta_1(s),s)$, *for all* $(t,s) \in [0,T]\times[0,T]$.

c) *For any* $c \in (0, \frac{T}{2})$, $\min_{t\in[c,T-c]} g_1(t,s) \geq \frac{c^{n-1}}{T^{n-1}} g_1(\theta_1(s),s)$, *for all* $s \in [0,T]$,

where $\theta_1(s) = s$ *if* $n = 2$ *and*

$$\theta_1(s) = \begin{cases} \frac{s}{1-(1-\frac{s}{T})^{\frac{n-1}{n-2}}}, & s \in (0,T], \\ \frac{T(n-2)}{n-1}, & s = 0, \end{cases} \quad \textit{if } n \geq 3.$$

Lemma 5. *Assume that* $d > 0$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $a_i \geq 0$ *for all* $i = 1,\ldots,p-2$. *Then the Green's function* G_1 *of the problem* (2.1)*–*(2.2) *is continuous on* $[0,T]\times[0,T]$ *and satisfies* $G_1(t,s) \geq 0$ *for all* $(t,s) \in [0,T]\times[0,T]$. *Moreover, if* $y \in C([0,T])$ *satisfies* $y(t) \geq 0$ *for all* $t \in [0,T]$, *then the unique solution* u *of problem* (2.1)*–*(2.2) *satisfies* $u(t) \geq 0$ *for all* $t \in [0,T]$.

Lemma 6. *Assume that* $d > 0$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $a_i \geq 0$ *for all* $i = 1,\ldots,p-2$. *Then the Green's function* G_1 *of problem* (2.1)*–*(2.2) *satisfies the inequalities*

a) $G_1(t,s) \le I_1(s)$, $\forall (t,s) \in [0,T] \times [0,T]$, *where*

$$I_1(s) = g_1(\theta_1(s), s) + \frac{T^{n-1}}{d} \sum_{i=1}^{p-2} a_i g_1(\xi_i, s), \quad \forall s \in [0,T];$$

b) *For every* $c \in (0, \frac{T}{2})$,

$$\min_{t\in[c,T-c]} G_1(t,s) \ge \frac{c^{n-1}}{T^{n-1}} I_1(s), \quad \forall s \in [0,T].$$

Lemma 7. *Assume that* $d > 0$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $a_i \ge 0$ *for all* $i = 1, \dots, p-2$, $c \in (0, \frac{T}{2})$ *and* $y \in C([0,T])$ *satisfies* $y(t) \ge 0$ *for all* $t \in [0,T]$. *Then the solution* $u(t)$, $t \in [0,T]$, *of problem* (2.1)–(2.2) *satisfies the inequality*

$$\min_{t\in[c,T-c]} u(t) \ge \frac{c^{n-1}}{T^{n-1}} \max_{t'\in[0,T]} u(t').$$

We can also formulate similar results as Lemmas 1–7 above for the boundary value problem

$$v^{(m)}(t) + h(t) = 0, \quad t \in (0,T), \tag{2.4}$$

$$v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(T) = \sum_{i=1}^{q-2} b_i v(\eta_i), \tag{2.5}$$

where $0 < \eta_1 < \cdots < \eta_{q-2} < T$, $b_i \ge 0$ for $i = 1, \dots, q-2$, and $h \in C([0,T])$. If $e = T^{m-1} - \sum_{i=1}^{q-2} b_i \eta_i^{m-1} \ne 0$, we denote by G_2 the Green's function associated to problem (2.4)–(2.5) and defined in a similar manner as G_1. We also denote by g_2, θ_2 and I_2 the corresponding functions for (2.4)–(2.5) defined in a similar manner as g_1, θ_1 and I_1, respectively.

3 Main Results

In this section, we shall investigate the existence of positive solutions for our problem (S)–(BC), under various assumptions on singular functions f and g.

We present the assumptions that we shall use in the sequel.

$(H1)$ $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $a_i \ge 0$, $i = 1, \dots, p-2$, $d = T^{n-1} - \sum_{i=1}^{p-2} a_i \xi_i^{n-1} > 0$, $0 < \eta_1 < \cdots < \eta_{q-2} < T$, $b_i \ge 0$, $i = 1, \dots, q-2$, $e = T^{m-1} - \sum_{i=1}^{q-2} b_i \eta_i^{m-1} > 0$.

$(H2)$ Functions $f, g \in C((0,T) \times \mathbb{R}_+, \mathbb{R}_+)$ and there exist $p_i \in C((0,T), \mathbb{R}_+)$, $q_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2$, with $0 < \int_0^T p_i(t)\,dt < \infty$, $i = 1, 2$, $q_1(0) = 0$, $q_2(0) = 0$ such that

$$f(t,x) \le p_1(t) q_1(x), \quad g(t,x) \le p_2(t) q_2(x), \quad \forall t \in (0,T),\ x \in \mathbb{R}_+.$$

$(H3)$ There exist $r_1, r_2 \in (0,\infty)$ with $r_1 r_2 \geq 1$ such that

$$\text{i)}\quad q_{10}^s = \limsup_{x\to 0^+} \frac{q_1(x)}{x^{r_1}} \in [0,\infty);\quad \text{ii)}\quad q_{20}^s = \limsup_{x\to 0^+} \frac{q_2(x)}{x^{r_2}} = 0.$$

$(H4)$ There exist $l_1, l_2 \in (0,\infty)$ with $l_1 l_2 \geq 1$ and $c \in (0, \frac{T}{2})$ such that

$$\text{i)}\quad f_\infty^i = \liminf_{x\to\infty} \inf_{t\in[c,T-c]} \frac{f(t,x)}{x^{l_1}} \in (0,\infty];$$
$$\text{ii)}\quad g_\infty^i = \liminf_{x\to\infty} \inf_{t\in[c,T-c]} \frac{g(t,x)}{x^{l_2}} = \infty.$$

$(H5)$ There exist $\alpha_1, \alpha_2 \in (0,\infty)$ with $\alpha_1\alpha_2 \leq 1$ such that

$$\text{i)}\quad q_{1\infty}^s = \limsup_{x\to \infty} \frac{q_1(x)}{x^{\alpha_1}} \in [0,\infty);\quad \text{ii)}\quad q_{2\infty}^s = \limsup_{x\to \infty} \frac{q_2(x)}{x^{\alpha_2}} = 0.$$

$(H6)$ There exist $\beta_1, \beta_2 \in (0,\infty)$ with $\beta_1\beta_2 \leq 1$ and $c \in (0, \frac{T}{2})$ such that

$$\text{i)}\quad f_0^i = \liminf_{x\to 0^+} \inf_{t\in[c,T-c]} \frac{f(t,x)}{x^{\beta_1}} \in (0,\infty];$$
$$\text{ii)}\quad g_0^i = \liminf_{x\to 0^+} \inf_{t\in[c,T-c]} \frac{g(t,x)}{x^{\beta_2}} = \infty.$$

The pair of functions $(u,v) \in (C([0,T]) \cap C^n((0,T))) \times (C([0,T]) \cap C^m((0,T)))$ is a solution for our problem (S)–(BC) if and only if $(u,v) \in C([0,T]) \times C([0,T])$ is a solution for the nonlinear integral equations

$$\begin{cases} u(t) = \displaystyle\int_0^T G_1(t,s) f\big(s, v(s)\big)\, ds, & t \in [0,T], \\ v(t) = \displaystyle\int_0^T G_2(t,s) g\big(s, u(s)\big)\, ds, & t \in [0,T]. \end{cases} \tag{3.1}$$

The system (3.1) can be written as the nonlinear integral system

$$\begin{cases} u(t) = \displaystyle\int_0^T G_1(t,s) f\Big(s, \int_0^T G_2(s,\tau) g\big(\tau, u(\tau)\big)\, d\tau\Big) ds, & t \in [0,T], \\ v(t) = \displaystyle\int_0^T G_2(t,s) g\big(s, u(s)\big)\, ds, & t \in [0,T]. \end{cases}$$

We consider the Banach space $X = C([0,T])$ with supremum norm $\|u\| = \sup_{t\in[0,T]} |u(t)|$ and define the cone $P \subset X$ by $P = \{u \in X,\ u(t) \geq 0,\ \forall t \in [0,T]\}$. For any $r > 0$, let

$$B_r = \{u \in C([0,T]),\ \|u\| < r\},\quad \partial B_r = \{u \in C([0,T]),\ \|u\| = r\}.$$

We also define the operator $\mathcal{A} : P \to X$ by

$$\mathcal{A}(u)(t) = \int_0^T G_1(t,s) f\Big(s, \int_0^T G_2(s,\tau) g\big(\tau, u(\tau)\big)\, d\tau\Big) ds.$$

Lemma 8. *Assume that $(H1)$–$(H2)$ hold. Then $\mathcal{A} : P \to P$ is completely continuous.*

Proof. We denote $\alpha = \int_0^T I_1(s)p_1(s)\,ds$ and $\beta = \int_0^T I_2(s)p_2(s)\,ds$. Using $(H2)$, we deduce that $0 < \alpha < \infty$ and $0 < \beta < \infty$. By Lemma 5 and the corresponding lemma for G_2, we get that $\mathcal{A}$ maps P into P.

We shall prove that $\mathcal{A}$ maps bounded sets into relatively compact sets. Suppose $D \subset P$ is an arbitrary bounded set. First we prove that $\mathcal{A}(D)$ is a bounded set. Because D is bounded, then there exists $M_1 > 0$ such that $\|u\| \le M_1$ for all $u \in D$. By the continuity of q_2, there exists $M_2 > 0$ such that $M_2 = \sup_{x\in[0,M_1]} q_2(x)$. By using Lemma 6 for G_2, for any $u \in D$ and $s \in [0, T]$, we obtain

$$\begin{aligned}\int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau &\le \int_0^T G_2(s,\tau)p_2(\tau)q_2\big(u(\tau)\big)\,d\tau\\ &\le M_2\int_0^T I_2(\tau)p_2(\tau)\,d\tau = \beta M_2. \qquad (3.2)\end{aligned}$$

Because q_1 is a continuous function, there exists $M_3 > 0$ such that $M_3 = \sup_{x\in[0,\beta M_2]} q_1(x)$. Therefore, from (3.2), $(H2)$ and Lemma 6, we deduce

$$\begin{aligned}(\mathcal{A}u)(t) &= \int_0^T G_1(t,s)f\Big(s,\int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\Big)ds\\ &\le \int_0^T G_1(t,s)p_1(s)q_1\Big(\int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\Big)ds\\ &\le M_3\int_0^T I_1(s)p_1(s)\,ds = \alpha M_3, \quad \forall t\in[0,T]. \qquad (3.3)\end{aligned}$$

So, $\|\mathcal{A}u\| \le \alpha M_3$ for all $u \in D$. Therefore $\mathcal{A}(D)$ is bounded.

In what follows, we shall prove that $\mathcal{A}(D)$ is equicontinuous. By using Lemma 3, we have

$$\begin{aligned}(\mathcal{A}u)(t) &= \int_0^T G_1(t,s)f\Big(s,\int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\Big)ds\\ &= \int_0^T \Big[g_1(t,s) + \frac{t^{n-1}}{d}\sum_{i=1}^{p-2} a_i g_1(\xi_i,s)\Big]f\Big(s,\int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\Big)ds\\ &= \frac{1}{(n-1)!T^{n-1}}\int_0^t \big[t^{n-1}(T-s)^{n-1} - T^{n-1}(t-s)^{n-1}\big]\\ &\quad\times f\Big(s,\int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\Big)ds\\ &\quad+ \frac{1}{(n-1)!T^{n-1}}\int_t^T t^{n-1}(T-s)^{n-1}f\Big(s,\int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\Big)ds\\ &\quad+ \frac{t^{n-1}}{d}\sum_{i=1}^{p-2} a_i\int_0^T g_1(\xi_i,s)f\Big(s,\int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\Big)ds, \quad \forall t\in[0,T].\end{aligned}$$

Therefore, for any $t \in (0,T)$, we obtain

$$\begin{aligned}
(\mathcal{A}u)'(t) &= \frac{t^{n-1}(T-t)^{n-1} - T^{n-1}(t-t)^{n-1}}{(n-1)!T^{n-1}} f\Big(t, \int_0^T G_2(t,\tau) g\big(\tau, u(\tau)\big)\, d\tau\Big) \\
&+ \int_0^t \frac{(n-1)t^{n-2}(T-s)^{n-1} - (n-1)T^{n-1}(t-s)^{n-2}}{(n-1)!T^{n-1}} \\
&\times f\Big(s, \int_0^T G_2(s,\tau) g\big(\tau, u(\tau)\big)\, d\tau\Big) ds \\
&- \frac{t^{n-1}(T-t)^{n-1}}{(n-1)!T^{n-1}} f\Big(t, \int_0^T G_2(t,\tau) g\big(\tau, u(\tau)\big)\, d\tau\Big) \\
&+ \int_t^T \frac{(n-1)t^{n-2}(T-s)^{n-1}}{(n-1)!T^{n-1}} f\Big(s, \int_0^T G_2(s,\tau) g\big(\tau, u(\tau)\big)\, d\tau\Big) \\
&+ \frac{(n-1)t^{n-2}}{d} \sum_{i=1}^{p-2} a_i \int_0^T g_1(\xi_i, s) f\Big(s, \int_0^T G_2(s,\tau) g\big(\tau, u(\tau)\big)\, d\tau\Big) ds.
\end{aligned}$$

So, for any $t \in (0,T)$, we deduce

$$\begin{aligned}
\big|(\mathcal{A}u)'(t)\big| &\le \int_0^t \frac{t^{n-2}(T-s)^{n-1} + T^{n-1}(t-s)^{n-2}}{(n-2)!T^{n-1}} p_1(s) \\
&\times g_1\Big(\int_0^T G_2(s,\tau) g(\tau, u(\tau))\, d\tau\Big) ds \\
&+ \int_t^T \frac{t^{n-2}(T-s)^{n-1}}{(n-2)!T^{n-1}} p_1(s) q_1\Big(\int_0^T G_2(s,\tau) g\big(\tau, u(\tau)\big)\, d\tau\Big)\, ds \\
&+ \frac{(n-1)t^{n-2}}{d} \sum_{i=1}^{p-2} a_i \int_0^T g_1(\xi_i, s) p_1(s) q_1\Big(\int_0^T G_2(s,\tau) g\big(\tau, u(\tau)\big)\, d\tau\Big) ds \\
&\le M_3\Big(\int_0^t \frac{t^{n-2}(T-s)^{n-1} + T^{n-1}(t-s)^{n-2}}{(n-2)!T^{n-1}} p_1(s)\, ds \\
&+ \int_t^T \frac{t^{n-2}(T-s)^{n-1}}{(n-2)!T^{n-1}} p_1(s)\, ds \\
&+ \frac{(n-1)t^{n-2}}{d} \sum_{i=1}^{p-2} a_i \int_0^T g_1(\xi_i, s) p_1(s)\, ds\Big).
\end{aligned} \tag{3.4}$$

We denote

$$\begin{aligned}
h(t) &= \int_0^t \frac{t^{n-2}(T-s)^{n-1} + T^{n-1}(t-s)^{n-2}}{(n-2)!T^{n-1}} p_1(s)\, ds \\
&\quad + \int_t^T \frac{t^{n-2}(T-s)^{n-1}}{(n-2)!T^{n-1}} p_1(s)\, ds, \\
\mu(t) &= h(t) + \frac{(n-1)t^{n-2}}{d} \sum_{i=1}^{p-2} a_i \int_0^T g_1(\xi_i, s) p_1(s)\, ds, \quad t \in (0,T).
\end{aligned}$$

For the integral of the function h, by exchanging the order of integration, we obtain

$$\begin{aligned}
\int_0^T h(t)\,dt &= \int_0^T \Big(\int_0^t \frac{t^{n-2}(T-s)^{n-1}+T^{n-1}(t-s)^{n-2}}{(n-2)!T^{n-1}} p_1(s)\,ds\Big)dt \\
&\quad + \int_0^T \Big(\int_t^T \frac{t^{n-2}(T-s)^{n-1}}{(n-2)!T^{n-1}} p_1(s)\,ds\Big)dt \\
&= \int_0^T \Big(\int_s^T \frac{t^{n-2}(T-s)^{n-1}+T^{n-1}(t-s)^{n-2}}{(n-2)!T^{n-1}} p_1(s)\,dt\Big)ds \\
&\quad + \int_0^T \Big(\int_0^s \frac{t^{n-2}(T-s)^{n-1}}{(n-2)!T^{n-1}} p_1(s)\,dt\Big)ds \\
&= \int_0^T \frac{(T-s)^{n-1}}{(n-2)!T^{n-1}} p_1(s)\Big(\int_s^T t^{n-2}\,dt\Big)ds + \int_0^T \frac{p_1(s)}{(n-2)!}\Big(\int_s^T (t-s)^{n-2}\,dt\Big)ds \\
&\quad + \int_0^T \frac{(T-s)^{n-1}}{(n-2)!T^{n-1}} p_1(s)\Big(\int_0^s t^{n-2}\,dt\Big)ds \\
&= \int_0^T \frac{(T-s)^{n-1}}{(n-2)!T^{n-1}} \Big(\frac{T^{n-1}-s^{n-1}}{n-1}\Big)p_1(s)\,ds + \int_0^T \frac{p_1(s)(T-s)^{n-1}}{(n-1)!}\,ds \\
&\quad + \int_0^T \frac{(T-s)^{n-1}s^{n-1}}{(n-1)!T^{n-1}} p_1(s)\,ds \\
&= \frac{2}{(n-1)!}\int_0^T (T-s)^{n-1}p_1(s)\,ds < \infty.
\end{aligned}$$

For the integral of the function μ, we have

$$\begin{aligned}
\int_0^T \mu(t)\,dt &= \int_0^T h(t)\,dt + \frac{n-1}{d}\sum_{i=1}^{p-2} a_i \int_0^T t^{n-2}\Big(\int_0^T g_1(\xi_i,s)p_1(s)\,ds\Big)dt \\
&\le \frac{2}{(n-1)!}\int_0^T (T-s)^{n-1}p_1(s)\,ds + \frac{T^{n-1}}{d}\sum_{i=1}^{p-2} a_i \int_0^T g_1\big(\theta_1(s),s\big)p_1(s)\,ds \\
&\le \frac{1}{(n-1)!}\big(2+\frac{T^{n-1}}{d}\sum_{i=1}^{p-2} a_i\big)\int_0^T (T-s)^{n-1}p_1(s)\,ds < \infty. \qquad (3.5)
\end{aligned}$$

We deduce that $\mu \in L^1(0,T)$. Thus for any given $t_1, t_2 \in [0,1]$ with $t_1 \le t_2$ and $u \in D$, by (3.4), we obtain

$$\big|(\mathcal{A}u)(t_1) - (\mathcal{A}u)(t_2)\big| = \Big|\int_{t_1}^{t_2} (\mathcal{A}u)'(t)\,dt\Big| \le M_3 \int_{t_1}^{t_2} \mu(t)\,dt. \qquad (3.6)$$

From (3.5), (3.6) and absolute continuity of the integral function, we obtain that $\mathcal{A}(D)$ is equicontinuous. This conclusion together with (3.3) and Ascoli-Arzelà theorem yields that $\mathcal{A}(D)$ is relatively compact. Therefore $\mathcal{A}$ is a compact operator.

By using similar arguments as those used in the proof of Lemma 2.4 from [12], we can show that $\mathcal{A}$ is continuous on P. Therefore $\mathcal{A} : P \to P$ is completely continuous. □

For $c \in (0, T/2)$, we define the cone

$$P_0 = \Big\{ u \in X,\ u(t) \geq 0,\ \forall t \in [0,T],\ \min_{t \in [c, T-c]} u(t) \geq \gamma \|u\| \Big\},$$

where $\gamma = \min\{c^{n-1}/T^{n-1}, c^{m-1}/T^{m-1}\}$. Under assumptions $(H1)$, $(H2)$, we have $\mathcal{A}(P) \subset P_0$. Indeed, for $u \in P$, let $v = \mathcal{A}(u)$. By Lemma 7, we have

$$\min_{t \in [c, T-c]} v(t) \geq \frac{c^{n-1}}{T^{n-1}} \|v\| \geq \gamma \|v\|,$$

that is $v \in P_0$.

Theorem 2. *Assume that $(H1)$–$(H4)$ hold. Then the problem (S)–(BC) has at least one positive solution $(u(t), v(t))$, $t \in [0,T]$.*

Proof. We consider the cone P_0 with c given in $(H4)$. From $(H3)$ i) and $(H2)$, we deduce that there exists $C_1 > 0$ such that

$$q_1(x) \leq C_1 x^{r_1}, \quad \forall x \in [0,1]. \tag{3.7}$$

From $(H3)$ ii) and $(H2)$, for $C_2 = \min\{(1/(C_1 \alpha \beta^{r_1}))^{1/r_1}, 1/\beta\} > 0$, we deduce that there exists $\delta_1 \in (0,1)$ such that

$$q_2(x) \leq C_2 x^{r_2}, \quad \forall x \in [0, \delta_1]. \tag{3.8}$$

From (3.8), $(H2)$ and Lemma 6, for any $u \in \partial B_{\delta_1} \cap P_0$ and $s \in [0,T]$, we obtain

$$\begin{aligned}
\int_0^T G_2(s,\tau) g\big(\tau, u(\tau)\big)\, d\tau &\leq \int_0^T G_2(s,\tau) p_2(\tau) q_2\big(u(\tau)\big)\, d\tau \\
&\leq C_2 \int_0^T G_2(s,\tau) p_2(\tau) \big(u(\tau)\big)^{r_2}\, d\tau \\
&\leq C_2 \int_0^T I_2(\tau) p_2(\tau)\, d\tau \cdot \|u\|^{r_2} = C_2 \beta \delta_1^{r_2} \leq \delta_1^{r_2} < 1.
\end{aligned} \tag{3.9}$$

By using (3.7)–(3.9) and $(H2)$, for any $u \in \partial B_{\delta_1} \cap P_0$ and $t \in [0,T]$, we obtain

$$\begin{aligned}
(\mathcal{A}u)(t) &\leq \int_0^T G_1(t,s) p_1(s) q_1\Big(\int_0^T G_2(s,\tau) g\big(\tau, u(\tau)\big)\, d\tau \Big) ds \\
&\leq C_1 \int_0^T G_1(t,s) p_1(s) \Big(\int_0^T G_2(s,\tau) g\big(\tau, u(\tau)\big)\, d\tau \Big)^{r_1} ds \\
&\leq C_1 \int_0^T G_1(t,s) p_1(s) \Big(\int_0^T G_2(s,\tau) p_2(\tau) q_2\big(u(\tau)\big)\, d\tau \Big)^{r_1} ds \\
&\leq C_1 \int_0^T G_1(t,s) p_1(s) \Big(C_2 \int_0^T G_2(s,\tau) p_2(\tau) \big(u(\tau)\big)^{r_2}\, d\tau \Big)^{r_1} ds \\
&\leq C_1 \int_0^T I_1(s) p_1(s) \Big(C_2 \int_0^T I_2(\tau) p_2(\tau) \big(u(\tau)\big)^{r_2}\, d\tau \Big)^{r_1} ds
\end{aligned}$$

$$\le C_1 \int_0^T I_1(s)p_1(s)\,ds \cdot \Big(C_2 \int_0^T I_2(\tau)p_2(\tau)\,d\tau\Big)^{r_1} \cdot \|u\|^{r_1 r_2}$$
$$\le C_1\alpha(C_2\beta)^{r_1}\|u\|^{r_1r_2} = C_1C_2^{r_1}\alpha\beta^{r_1}\|u\|^{r_1r_2} \le \|u\|^{r_1r_2} \le \|u\|.$$

Therefore

$$\|\mathcal{A}u\| \le \|u\|, \quad \forall u \in \partial B_{\delta_1} \cap P_0. \tag{3.10}$$

From $(H4)$ i), we deduce that there exist $C_3 > 0$ and $x_1 > 0$ such that

$$f(t,x) \ge C_3 x^{l_1}, \quad \forall x \ge x_1,\ \forall t \in [c, T-c]. \tag{3.11}$$

We consider now

$$C_4 = \max\Big\{\frac{T^{m-1}}{c^{m-1}\gamma^{l_2}\theta_2}, \Big(\frac{T^{l_1(m-1)+n-1}}{c^{l_1(m-1)+n-1}C_3\gamma^{l_1l_2}\theta_1\theta_2^{l_1}}\Big)^{1/l_1}\Big\} > 0,$$

where $\theta_1 = \int_c^{T-c} I_1(s)\,ds > 0$ and $\theta_2 = \int_c^{T-c} I_2(s)\,ds > 0$. From $(H4)$ ii), we deduce that there exists $x_2 \ge 1$ such that

$$g(t,x) \ge C_4 x^{l_2}, \quad \forall x \ge x_2,\ \forall t \in [c, T-c]. \tag{3.12}$$

Now we choose $R_0 = \max\{x_1, x_2\}$ and $R > \max\{R_0/\gamma, R_0^{1/l_2}\}$. Then for any $u \in \partial B_R \cap P_0$, we have $\min_{t\in[c,T-c]} u(t) \ge \gamma\|u\| = \gamma R > R_0$. By using (3.11) and (3.12), for any $u \in \partial B_R \cap P_0$ and $s \in [c, T-c]$, we obtain

$$\int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau \ge \int_c^{T-c} G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau$$
$$\ge \frac{c^{m-1}}{T^{m-1}}\int_c^{T-c} I_2(\tau)C_4\big(u(\tau)\big)^{l_2}\,d\tau \ge \frac{c^{m-1}}{T^{m-1}}C_4\gamma^{l_2}\int_c^{T-c} I_2(\tau)\,d\tau \cdot \|u\|^{l_2}$$
$$\ge \|u\|^{l_2} = R^{l_2} > R_0.$$

Then for any $u \in \partial B_R \cap P_0$ and $t \in [c, T-c]$, we have

$$(\mathcal{A}u)(t) = \int_0^T G_1(t,s)f\Big(s, \int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\Big)ds$$
$$\ge \int_c^{T-c} G_1(t,s)f\Big(s, \int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\Big)ds$$
$$\ge C_3\int_c^{T-c} G_1(t,s)\Big(\int_c^{T-c} G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\Big)^{l_1}ds$$
$$\ge C_3\int_c^{T-c} G_1(t,s)\Big(\frac{c^{m-1}}{T^{m-1}}\int_c^{T-c} I_2(\tau)C_4\big(u(\tau)\big)^{l_2}\,d\tau\Big)^{l_1}ds$$
$$\ge C_3C_4^{l_1}\frac{c^{l_1(m-1)}}{T^{l_1(m-1)}}\int_c^{T-c} G_1(t,s)\gamma^{l_1l_2}\|u\|^{l_1l_2}\Big(\int_c^{T-c} I_2(\tau)\,d\tau\Big)^{l_1}ds$$
$$\ge C_3C_4^{l_1}\frac{c^{l_1(m-1)+n-1}}{T^{l_1(m-1)+n-1}}\gamma^{l_1l_2}\Big(\int_c^{T-c} I_1(s)\,ds\Big)\Big(\int_c^{T-c} I_2(\tau)\,d\tau\Big)^{l_1}\|u\|^{l_1l_2}$$
$$\ge \|u\|^{l_1l_2} \ge \|u\|.$$

Therefore we obtain

$$\|\mathcal{A}u\| \geq \|u\|, \quad \forall u \in \partial B_R \cap P_0. \tag{3.13}$$

By (3.10), (3.13) and Theorem 1 i), we obtain that $\mathcal{A}$ has a fixed point $u_1 \in (\bar{B}_R \setminus B_{\delta_1}) \cap P_0$, that is $\delta_1 < \|u_1\| < R$. Let $v_1(t) = \int_0^T G_2(t,s)g(s,u_1(s))\,ds$. Then $(u_1, v_1) \in P_0 \times P_0$ is a solution of (S)–(BC). In addition $\|v_1\| > 0$. Indeed, if we suppose that $v_1(t) = 0$ for all $t \in [0,T]$, then by using $(H2)$ we have $f(s, v_1(s)) = f(s,0) = 0$ for all $s \in [0,T]$. This implies $u_1(t) = 0$ for all $t \in [0,T]$, which contradicts $\|u_1\| > 0$. By using Theorem 1.1 from [11] (see [1]), we obtain $u_1(t) > 0$ and $v_1(t) > 0$ for all $t \in (0, T-c]$. The proof of Theorem 2 is completed. □

Theorem 3. *Assume that* $(H1)$, $(H2)$, $(H5)$ *and* $(H6)$ *hold. Then the problem* (S)–(BC) *has at least one positive solution* $(u(t), v(t))$, $t \in [0,T]$.

Proof. We consider the cone P_0 with c given in $(H6)$. By $(H5)$ i) we deduce that there exist $C_5 > 0$ and $C_6 > 0$ such that

$$q_1(x) \leq C_5 x^{\alpha_1} + C_6, \quad \forall x \in [0,\infty). \tag{3.14}$$

From $(H5)$ ii), for $\varepsilon_0 > 0$, $\varepsilon_0 < (2^{\alpha_1} C_5 \alpha \beta^{\alpha_1})^{-1/\alpha_1}$, we deduce that there exists $C_7 > 0$ such that

$$q_2(x) \leq \varepsilon_0 x^{\alpha_2} + C_7, \quad \forall x \in [0,\infty). \tag{3.15}$$

By using (3.14), (3.15) and $(H2)$, for any $u \in P_0$, we obtain

$$\begin{aligned}
(\mathcal{A}u)(t) &\leq \int_0^T G_1(t,s)p_1(s)q_1\Big(\int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\Big)ds \\
&\leq \int_0^T G_1(t,s)p_1(s)\Big[C_5\Big(\int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\Big)^{\alpha_1} + C_6\Big]ds \\
&\leq C_5 \int_0^T G_1(t,s)p_1(s)\Big(\int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\Big)^{\alpha_1} ds \\
&\quad + C_6 \int_0^T I_1(s)p_1(s)\,ds \\
&\leq C_5 \int_0^T I_1(s)p_1(s)\Big[\int_0^T G_2(s,\tau)p_2(\tau)\big(\varepsilon_0\big(u(\tau)\big)^{\alpha_2} + C_7\big)\,d\tau\Big]^{\alpha_1} ds + \alpha C_6 \\
&\leq C_5 \int_0^T I_1(s)p_1(s)\,ds\Big(\int_0^T I_2(\tau)p_2(\tau)\,d\tau\Big)^{\alpha_1}\big(\varepsilon_0\|u\|^{\alpha_2} + C_7\big)^{\alpha_1} + \alpha C_6 \\
&= C_5 \alpha \beta^{\alpha_1}\big(\varepsilon_0\|u\|^{\alpha_2} + C_7\big)^{\alpha_1} + \alpha C_6 \\
&\leq 2^{\alpha_1} C_5 \alpha \beta^{\alpha_1}\big(\varepsilon_0^{\alpha_1}\|u\|^{\alpha_1\alpha_2} + C_7^{\alpha_1}\big) + \alpha C_6 \\
&= C_5 2^{\alpha_1}\varepsilon_0^{\alpha_1}\alpha\beta^{\alpha_1}\|u\|^{\alpha_1\alpha_2} + C_5 2^{\alpha_1}\alpha\beta^{\alpha_1}C_7^{\alpha_1} + \alpha C_6, \quad \forall t \in [0,T].
\end{aligned}$$

By definition of ε_0, we can choose sufficiently large $R_1 > 0$ such that

$$\|\mathcal{A}u\| \leq \|u\|, \quad \forall u \in \partial B_{R_1} \cap P_0. \tag{3.16}$$

From $(H6)$ i), we deduce that there exist positive constants $C_8 > 0$ and $x_3 > 0$ such that $f(t,x) \geq C_8 x^{\beta_1}$, for all $x \in [0, x_3]$ and $t \in [c, T-c]$. From $(H6)$ ii), for $\varepsilon_1 = (\frac{T^{n-1+\beta_1(m-1)}}{C_8 c^{n-1+\beta_1(m-1)}\gamma^{\beta_1\beta_2}\theta_1\theta_2^{\beta_1}})^{1/\beta_1} > 0$, we deduce that there exists $x_4 > 0$ such that $g(t,x) \geq \varepsilon_1 x^{\beta_2}$ for all $x \in [0, x_4]$ and $t \in [c, T-c]$.

We consider $x_5 = \min\{x_3, x_4\}$. So we obtain

$$f(t,x) \geq C_8 x^{\beta_1}, \quad g(t,x) \geq \varepsilon_1 x^{\beta_2}, \ \forall (t,x) \in [c, T-c] \times [0, x_5]. \tag{3.17}$$

From assumption $q_2(0) = 0$ and the continuity of q_2, we deduce that there exists sufficiently small $\varepsilon_2 \in (0, \min\{x_5, 1\})$ such that $q_2(x) \leq \beta^{-1} x_5$ for all $x \in [0, \varepsilon_2]$. Therefore for any $u \in \partial B_{\varepsilon_2} \cap P_0$ and $s \in [0, T]$, we have

$$\begin{aligned}\int_0^T G_2(s,\tau) g\big(\tau, u(\tau)\big)\, d\tau &\leq \int_0^T G_2(s,\tau) p_2(\tau) q_2\big(u(\tau)\big)\, d\tau \\ &\leq \beta^{-1} x_5 \int_0^T I_2(\tau) p_2(\tau)\, d\tau = x_5. \end{aligned}\tag{3.18}$$

By (3.17), (3.18), Lemma 6 and Lemma 7, for any $t \in [c, T-c]$, we have

$$\begin{aligned}(\mathcal{A}u)(t) &\geq \int_c^{T-c} G_1(t,s) f\Big(s, \int_0^T G_2(s,\tau) g\big(\tau, u(\tau)\big)\, d\tau\Big) ds \\ &\geq C_8 \int_c^{T-c} G_1(t,s) \Big(\int_c^{T-c} G_2(s,\tau) g\big(\tau, u(\tau)\big)\, d\tau\Big)^{\beta_1} ds \\ &\geq C_8 \int_c^{T-c} G_1(t,s) \Big(\varepsilon_1 \int_c^{T-c} G_2(s,\tau) \big(u(\tau)\big)^{\beta_2}\, d\tau\Big)^{\beta_1} ds \\ &\geq \frac{C_8 c^{n-1}}{T^{n-1}} \int_c^{T-c} I_1(s) \Big[\Big(\frac{\varepsilon_1 c^{m-1}}{T^{m-1}}\Big)^{\beta_1} \Big(\int_c^{T-c} I_2(\tau) \big(u(\tau)\big)^{\beta_2}\, d\tau\Big)^{\beta_1}\Big] ds \\ &\geq \frac{C_8 c^{n-1+(m-1)\beta_1} \varepsilon_1^{\beta_1} \gamma^{\beta_1\beta_2} \theta_1 \theta_2^{\beta_1}}{T^{n-1+(m-1)\beta_1}} \|u\|^{\beta_1\beta_2} = \|u\|^{\beta_1\beta_2} \geq \|u\|. \end{aligned}$$

Therefore

$$\|\mathcal{A}u\| \geq \|u\|, \quad \forall u \in \partial B_{\varepsilon_2} \cap P_0. \tag{3.19}$$

By (3.16), (3.19) and Theorem 1 ii), we deduce that $\mathcal{A}$ has at least one fixed point $u_2 \in (\bar{B}_{R_1} \setminus B_{\varepsilon_2}) \cap P_0$. Then our problem (S)–(BC) has at least one positive solution $(u_2, v_2) \in P_0 \times P_0$ where $v_2(t) = \int_0^T G_2(t,s) g(s, u(s))\, ds$. The proof of Theorem 3 is completed. □

4 Examples

In this section, we shall present two examples which illustrate our results.

Example 1. Let

$$f(t,x) = \frac{x^a}{t^{\gamma_1}(T-t)^{\delta_1}}, \quad g(t,x) = \frac{x^b}{t^{\gamma_2}(T-t)^{\delta_2}},$$

with $a, b > 1$ and $\gamma_1, \delta_1, \gamma_2, \delta_2 \in (0,1)$. Here $f(t,x) = p_1(t)q_1(x)$ and $g(t,x) = p_2(t)q_2(x)$, where

$$p_1(t) = \frac{1}{t^{\gamma_1}(T-t)^{\delta_1}}, \quad p_2(t) = \frac{1}{t^{\gamma_2}(T-t)^{\delta_2}}, \quad q_1(x) = x^a, \quad q_2(x) = x^b.$$

We have $0 < \int_0^T p_1(s)\,ds < \infty$, $0 < \int_0^T p_2(s)\,ds < \infty$.

In $(H3)$, for $r_1 < a$, $r_2 < b$ and $r_1 r_2 \geq 1$, we have

$$\limsup_{x\to 0^+} \frac{q_1(x)}{x^{r_1}} = \lim_{x\to 0^+} x^{a-r_1} = 0, \qquad \limsup_{x\to 0^+} \frac{q_2(x)}{x^{r_2}} = \lim_{x\to 0^+} x^{b-r_2} = 0.$$

In $(H4)$, for $l_1 < a$, $l_2 < b$, $l_1 l_2 \geq 1$ and $c \in (0, \frac{T}{2})$, we have

$$\begin{aligned} &\liminf_{x\to\infty} \inf_{t\in[c,T-c]} \frac{f(t,x)}{x^{l_1}} = \liminf_{x\to\infty} \inf_{t\in[c,T-c]} \frac{x^{a-l_1}}{t^{\gamma_1}(T-t)^{\delta_1}} \\ &= \left(\max\left\{ \frac{\gamma_1^{\gamma_1}\delta_1^{\delta_1}T^{\gamma_1+\delta_1}}{(\gamma_1+\delta_1)^{\gamma_1+\delta_1}}, c^{\gamma_1}(T-c)^{\delta_1}, c^{\delta_1}(T-c)^{\gamma_1} \right\} \right)^{-1} \lim_{x\to\infty} x^{a-l_1} = \infty. \end{aligned}$$

In a similar manner, we have

$$\liminf_{x\to\infty} \inf_{t\in[c,T-c]} \frac{g(t,x)}{x^{l_2}} = \infty.$$

For example, if $a = 2$, $b = 3/2$, $r_1 = 1$, $r_2 = 4/3$, $l_1 = 3/2$, $l_2 = 1$, the above conditions are satisfied. Under the assumption $(H1)$, by Theorem 2, we deduce that the problem (S)–(BC) has at least one positive solution.

Example 2. Let $f(t,x) = \frac{x^a(2+\cos x)}{t^\gamma}$, $g(t,x) = \frac{x^b(1+\sin x)}{(T-t)^\delta}$, with $a, b \in (0,1)$ and $\gamma, \delta \in (0,1)$. Here $f(t,x) = p_1(t)q_1(x)$ and $g(t,x) = p_2(t)q_2(x)$, where

$$p_1(t) = \frac{1}{t^\gamma}, \quad p_2(t) = \frac{1}{(T-t)^\delta}, \quad q_1(x) = x^a(2+\cos x), \quad q_2(x) = x^b(1+\sin x).$$

We have $0 < \int_0^T p_1(s)\,ds < \infty$, $0 < \int_0^T p_2(s)\,ds < \infty$.

In $(H5)$, for $\alpha_1 = a$, $\alpha_2 > b$ and $\alpha_1\alpha_2 \leq 1$, we have

$$\begin{aligned} \limsup_{x\to\infty} \frac{q_1(x)}{x^{\alpha_1}} &= \limsup_{x\to\infty} \frac{x^a(2+\cos x)}{x^{\alpha_1}} = 3, \\ \limsup_{x\to\infty} \frac{q_2(x)}{x^{\alpha_2}} &= \limsup_{x\to\infty} \frac{x^b(1+\sin x)}{x^{\alpha_2}} = 0. \end{aligned}$$

In $(H6)$, for $\beta_1 = a$, $\beta_2 > b$, $\beta_1\beta_2 \leq 1$ and $c \in (0, \frac{T}{2})$, we have

$$\begin{aligned} \liminf_{x\to 0^+} \inf_{t\in[c,T-c]} \frac{f(t,x)}{x^{\beta_1}} &= \liminf_{x\to 0^+} \inf_{t\in[c,T-c]} \frac{x^a(2+\cos x)}{t^\gamma x^{\beta_1}} = \frac{3}{(T-c)^\gamma} > 0, \\ \liminf_{x\to 0^+} \inf_{t\in[c,T-c]} \frac{g(t,x)}{x^{\beta_2}} &= \liminf_{x\to 0^+} \inf_{t\in[c,T-c]} \frac{x^b(1+\sin x)}{(T-t)^\delta x^{\beta_2}} \\ &= \frac{1}{(T-c)^\delta} \lim_{x\to 0^+} x^{b-\beta_2} = \infty. \end{aligned}$$

For example, if $a = 1/3$, $b = 1/2$, $\alpha_1 = 1/3$, $\alpha_2 = 1$, $\beta_1 = 1/3$, $\beta_2 = 1$, the above conditions are satisfied. Under the assumption $(H1)$, by Theorem 3, we deduce that the problem (S)–(BC) has at least one positive solution.

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