

Growth Models with Oblique Asymptote

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Abstract. A class of smooth functions which can be used as regression models for modelling phenomena requiring an oblique asymptote is analyzed. These types of models were defined as a product of a linear function and some well known growth models. In addition to their increasing character with an oblique asymptote, the resulting models provide curves with a single inflection point.

Keywords: growth models, oblique asymptote, single inflection point.

AMS Subject Classification: 26A48; 65D10; 62J02.

1 Introduction

Growth curve modelling using functions with an oblique asymptote and one or several inflection points are of interest in practice. Indeed, in economy smooth convex increasing curves with an oblique asymptote were considered in [30] to establish a relationship between household income and expenditure. In hydrology, for fitting the rating curves, it could be useful to have growth functions with one or more than one inflection point (see the U.S. Geological Survey's National Streamflow Information Program web page for more details). Also modelling with a varying carrying capacity leads to growth curve with more than one inflection point (see [16, 17, 29]).

In various fields, growth functions with an horizontal asymptote, such as Michaelis–Menten, Richards, Gompertz, Logistic, and Bridge functions, are commonly and widely used, e.g. agriculture [9, 10, 11, 12, 18], biology [1, 8, 19, 20, 31], economy [27], engineering [26], fishery [25], forestry [22, 33, 35], hydrology [2, 3, 5, 6, 7, 13, 21, 32], medicine [15], and other areas of applied research [23, 24, 28]. These growth functions are characterized by an horizontal asymptote and a single inflection point [4]. Other growth functions with these characteristics can be found in the literature (see [9, 14, 34] for example).

In this paper we extend our previous study about the already mentioned models [4, 5, 6]. We modify them to obtain models with an oblique asymptote while keeping the single inflection point property. The structure of the present

paper is as follows. In Section 2 we start with a general study of our modified model and obtain its main properties. In Section 3, using the analysis done in the preceding section, we modify some growth functions having a horizontal asymptote to obtain models having an increasing linear asymptote. Finally in Section 4 we present some numerical examples on synthetic data.

2 Main Results

Let us start with a general growth model $f(t)$, defined and twice continuously differentiable on an interval $(T_0, +\infty)$, and having the following basic properties:

- $f(t) \geq 0$ is an increasing function on $(T_0, +\infty)$,
- f has one and only one inflection point on $(T_0, +\infty)$ at $(t^*, f^* = f(t^*))$,
- $\lim_{t \rightarrow T_0^+} f(t) = f_{T_0^+} \geq 0, \quad \lim_{t \rightarrow +\infty} f(t) = f_{+\infty} > f_{T_0^+}$.

Let us assume that the second derivative of this growth functions can be decomposed as follows

$$f^{(2)}(t) = h(t)f^{(1)}(t), \tag{2.1}$$

where

$$f^{(1)}(t) > 0 \quad \text{for all } T_0 < t < +\infty,$$

$$h(t) \begin{cases} > 0 & \text{if } T_0 < t < t^*, \\ = 0 & \text{if } t = t^*, \\ < 0 & \text{if } t > t^*. \end{cases}$$

Remember that the Michaelis–Menten, Richards (generalized Logistic), Gompertz, Logistic, and Bridge models given in Table 1 have these properties, see [4].

In order to realize our goal of obtaining a growth function with an oblique asymptote, we multiply the growth function $f(t)$ by a linear increasing function given by $m(t) = pt + q$ with $p > 0$ and $q \geq 0$ to obtain

$$F(t) = m(t)f(t).$$

Moreover the zero of $m(t)$, given by $-q/p$ is required to be much less than t^* , the abscissa of the inflection point of $f(t)$, then $-q/p < t^*$. We are going to consider $F(t)$ for $t > \max \{-q/p, T_0\} = \bar{T}_0$. Then we have

$$F(t) = m(t)f(t) = [pt + q]f(t) \geq 0, \quad F^{(1)}(t) = pf(t) + m(t)f^{(1)}(t) > 0.$$

Hence $F(t)$ is a positive increasing function for $t > \bar{T}_0$. Also, from (2.1),

$$F^{(2)}(t) = 2pf^{(1)}(t) + m(t)f^{(2)}(t) = [2p + h(t)(pt + q)]f^{(1)}(t), \tag{2.2}$$

and it follows that

$$F^{(2)}(t) > 0 \quad \text{for all } \bar{T}_0 < t \leq t^*.$$

Table 1. Parameter values and inflection point of the growth models.

Model	Function	Parameters	Inflection point (t^*, f^*)
Menten	$f_M(t) = \frac{\omega_0 k^c + \omega_f t^c}{k^c + t^c}$	$k > 0, c > 1,$ $\omega_f > \omega_0 \geq 0$	$\left(k\left(\frac{c-1}{c+1}\right)^{1/c}, \frac{\omega_0 + \omega_f}{2} - \frac{\omega_f - \omega_0}{2c}\right)$
Richards ⁽¹⁾	$f_R(t) = a(1 - be^{-ct})^m$	$a > 0, b > 0,$ $c > 0, m > 1$	$\left(\frac{\ln(mb)}{c}, a\left(1 - \frac{1}{m}\right)^m\right)$
Richards ⁽²⁾	$f_R(t) = a(1 - be^{-ct})^m$	$a > 0, b < 0,$ $c > 0, m < 0$	$\left(\frac{\ln(mb)}{c}, a\left(1 - \frac{1}{m}\right)^m\right)$
Gompertz	$f_G(t) = ae^{-be^{-ct}}$	$a > 0, b > 0, c > 0$	$\left(\frac{\ln(b)}{c}, ae^{-1}\right)$
Logistic	$f_L(t) = \frac{a}{1+be^{-ct}}$	$a > 0, b > 0, c > 0$	$\left(\frac{\ln(b)}{c}, \frac{a}{2}\right)$
Bridge	$f_B(t) = a(1 - e^{-mt^b})$	$a > 0, b > 1, m > 0$	$\left(\left(\frac{b-1}{mb}\right)^{1/b}, a\left(1 - e^{-\frac{b-1}{b}}\right)\right)$

Moreover, suppose we can write

$$2p + h(t)(pt + q) = Q(t)g(t),$$

where $g(t) > 0$ for $t \in (\bar{T}_0, +\infty)$. Then $Q(t) > 0$ for $t \in (\bar{T}_0, t^*]$, and

$$Q(t^*) = \frac{2p}{g(t^*)} > 0.$$

Since we have

$$F^{(2)}(t) = Q(t)g(t)f^{(1)}(t), \tag{2.3}$$

to show that $F(t)$ has one and only one inflection point, it remains to prove that the sign of $Q(t)$ changes only once on $(t^*, +\infty)$. Some conditions on the expression of $Q(t)$ which ensure this fact are given in the next two theorems.

Theorem 1. *Let $Q(t)$ be a function defined by*

$$Q(t) = (A + Bt) - (C + Dt)t^\gamma \quad \text{for } t \geq 0,$$

where $B > 0$ and $D > 0$ are two positive real numbers and $\gamma \geq 1$. If there exists a positive real number $t^* \geq 0$ such that $Q(t^*) \geq 0$, then there exists a unique $T^* > t^*$ such that

$$Q(t) \begin{cases} > 0 & \text{if } t^* \leq t < T^*, \\ = 0 & \text{if } t = T^*, \\ < 0 & \text{if } t > T^*. \end{cases} \tag{2.4}$$

Proof. Let us suppose that there exists a $t^* \geq 0$ such that $Q(t^*) > 0$. Under the assumptions on D and γ , we also have

$$\lim_{t \rightarrow +\infty} Q(t) = -\infty.$$

For the first derivative of $Q(t)$, we have

$$\begin{aligned} Q^{(1)}(t) &= B - [D(\gamma + 1)t + \gamma C]t^{\gamma-1} \\ &= \left\{ \frac{1}{t^{\gamma-1}} - \left[\frac{D(\gamma + 1)t}{B} + \frac{\gamma C}{B} \right] \right\} Bt^{\gamma-1}. \end{aligned}$$

In the above equation the linear factor in brackets is strictly increasing since B and D are strictly positive. So to determine the sign of the term between braces we have to study two cases.

Case $\gamma > 1$: the function $t \mapsto 1/t^{\gamma-1}$ is strictly decreasing, so there exists a unique $\tau > 0$ such that $Q^{(1)}(\tau) = 0$, and then

$$Q^{(1)}(t) \begin{cases} > 0 & \text{if } 0 < t < \tau, \\ = 0 & \text{if } t = \tau, \\ < 0 & \text{if } t > \tau. \end{cases}$$

Case $\gamma = 1$: the existence of $\tau > 0$ such that $Q^{(1)}(\tau) = 0$ requires that $C < B$, otherwise we have $Q^{(1)}(t) < 0$ for all $t > 0$.

In both cases above, there exists a unique $T^* > t^*$ such that the condition (2.4) holds. \square

Remark. We can allow $D = 0$ in Theorem 1 and the conclusion holds under the following conditions :

- If $\gamma = 1$, we have $Q(t) = A + (B - C)t$. In this case, to ensure that the condition (2.4) holds we must have $B < C$, because otherwise for all $t \geq t^*$ we have $Q(t) \geq Q(t^*) > 0$.
- If $\gamma > 1$, we must have $C > 0$ because otherwise $Q^{(1)}(t) > 0$ for all $t \geq t^*$.

Theorem 2. *Let $Q(t)$ be a function defined by*

$$Q(t) = (A - Bt) + (C + Dt)e^{-\gamma t} \quad \text{for } t \in \mathbb{R},$$

where $B > 0$ and $D \geq 0$ are two positive real numbers and $\gamma > 0$. Suppose there exists a real number t^* such that $Q(t^*) > 0$, then there exists a unique $T^* > t^*$ such that

$$Q(t) \begin{cases} > 0 & \text{if } t^* \leq t < T^*, \\ = 0 & \text{if } t = T^*, \\ < 0 & \text{if } t > T^*. \end{cases} \tag{2.5}$$

Proof. Let us assume that there exists $t^* \in \mathbb{R}$ such that $Q(t^*) > 0$. Under the assumptions on B and γ we also have

$$\lim_{t \rightarrow +\infty} Q(t) = -\infty.$$

For the first derivative of $Q(t)$, we have

$$\begin{aligned} Q^{(1)}(t) &= -B + [D - \gamma(C + Dt)]e^{-\gamma t} \\ &= \left\{ -e^{\gamma t} + \left[\frac{D - \gamma C}{B} - \frac{\gamma D}{B}t \right] \right\} B e^{-\gamma t}. \end{aligned}$$

In the above equation the linear factor in brackets is strictly decreasing since B , D , and γ are strictly positive (or a constant if $D = 0$). Moreover, the exponential function $t \mapsto e^{\gamma t}$ is strictly increasing with $\lim_{t \rightarrow -\infty} e^{\gamma t} = 0$ and $\lim_{t \rightarrow +\infty} e^{\gamma t} = +\infty$. Thus, there exists one and only one $\tau \in \mathbb{R}$ such that $Q^{(1)}(\tau) = 0$ and then

$$Q^{(1)}(t) \begin{cases} > 0 & \text{if } t < \tau, \\ = 0 & \text{if } t = \tau, \\ < 0 & \text{if } t > \tau. \end{cases}$$

Hence the function $Q(t)$ is strictly increasing for $t \leq \tau$ and strictly decreasing for $t \geq \tau$. Since $Q(t^*) > 0$, there exists a unique $T^* > t^*$ such that the condition (2.5) holds. \square

From the two theorems, we have

$$F^{(2)}(t) \begin{cases} > 0 & \text{if } \bar{T}_0 < t < T^*, \\ = 0 & \text{if } t = T^*, \\ < 0 & \text{if } t > T^*. \end{cases} \quad (2.6)$$

The function $F(t)$ is a positive strictly increasing function, convex on (\bar{T}_0, T^*) , concave on $(T^*, +\infty)$, with a single inflection point at $(T^*, F^* = F(T^*))$. Moreover the oblique asymptote is given by $y = [pt + q]f_{+\infty}$.

In the next section, we apply these previous ideas on the six models mentioned in Table 1. This study does not cover all the existing models used in practice (see [11,12,14,23,34,35]), but any other model with a sigmoidal behavior can be modified in a similar way with the use of Theorem 1 or Theorem 2.

3 Models

The basic models are given in Table 1. We use them and consider the product

$$F(t) = m(t)f(t) = [pt + q]f(t) \quad (3.1)$$

to obtain the modified models listed in Table 2. Let us observe that the multiplicative constants, ω_f or a , in the basic models are included in the parameters p and q of the modified models, and $f_{+\infty} = 1$. Using the decomposition (2.3), we look for a unique $T^* > t^*$ such that

$$Q(t) \begin{cases} > 0 & \text{if } t^* \leq t < T^*, \\ = 0 & \text{if } t = T^*, \\ < 0 & \text{if } t > T^*, \end{cases} \quad (3.2)$$

to obtain the desired shape for the graph of $F(t)$ for the values of the parameters (p, q, θ) in the feasible set Θ specified in the last two columns of the Table 2. The graphs of f and F for all these models look like the graphs given in Figure 1.

Table 2. Growth models with an oblique asymptote $y = pt + q$.

Model	Function	Parameters $(p, q, \theta) \in \Theta$	
	$F(t) = [pt + q]f(t)$	p, q	θ
Michaelis–Menten	$F_M(t) = [pt + q] \frac{\omega_0 k^c + t^c}{k^c + t^c}$	$p > 0, q \geq 0$	$k > 0, c > 1, 0 \leq \omega_0 < 1$
Richards ⁽¹⁾	$F_R(t) = [pt + q](1 - be^{-ct})^m$	$p > 0, q \geq 0$	$b > 0, c > 0, m > 1$
Richards ⁽²⁾	$F_R(t) = [pt + q](1 - be^{-ct})^m$	$p > 0, q \geq 0$	$b < 0, c > 0, m < 0$
Gompertz	$F_G(t) = [pt + q]e^{-be^{-ct}}$	$p > 0, q \geq 0$	$b > 0, c > 0$
Logistic	$F_L(t) = \frac{[pt+q]}{1+be^{-ct}}$	$p > 0, q \geq 0$	$b > 0, c > 0$
Bridge	$F_B(t) = [pt + q](1 - e^{-mt^b})$	$p > 0, q \geq 0$	$b > 1, m > 0$

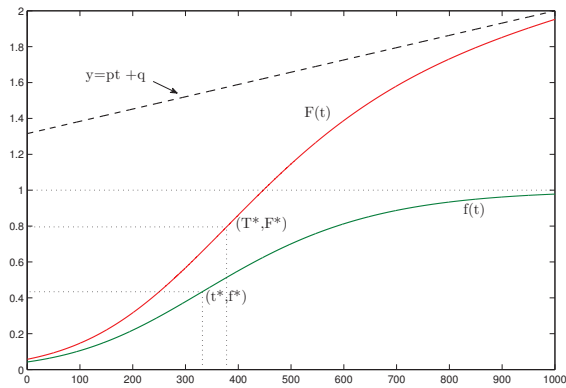


Figure 1. Graphic representation of a growth model $f(t)$ and its modified model $F(t)$ with their respective asymptotes $y = 1$ and $y = pt + q$ and inflection points (t^*, f^*) and (T^*, F^*) .

3.1 Michaelis–Menten model $f_M(t) = (\omega_0 k^c + t^c)/(k^c + t^c)$

To ensure the increasing character of this function with the existence of an inflection point we assume that $k > 0, c \geq 1$ and $0 \leq \omega_0 < 1$. The Michaelis–Menten function is well defined when $t \geq T_0 = 0$. Moreover we have

$$f_M^{(1)}(t) = \frac{c(1 - \omega_0)k^c t^{c-1}}{(k^c + t^c)^2} \quad \text{and} \quad f_M^{(2)}(t) = [(c - 1)k^c - (c + 1)t^c] \frac{f_M^{(1)}(t)}{t(k^c + t^c)}.$$

From (3.1) and (2.2), it follows that

$$F_M^{(2)}(t) = [k^c(q(c - 1) + p(c + 1)t) - (q(c + 1) + p(c - 1)t)t^c] \frac{f_M^{(1)}(t)}{t(k^c + t^c)}.$$

Then we have

$$Q(t) = k^c(q(c - 1) + p(c + 1)t) - (q(c + 1) + p(c - 1)t)t^c,$$

and

$$g(t) = \frac{1}{t(k^c + t^c)}.$$

Thus, Theorem 1 with $B = (c + 1)p > 0$, $D = (c - 1)p \geq 0$ and $\gamma = c$ ensures the existence of a unique $T^* > t^*$ such that the condition (3.2) holds.

3.2 Richards model $f_R(t) = (1 - be^{-ct})^m$

The function is defined as long as $1 - be^{-ct} \geq 0$. In this case we have

$$f_R^{(1)}(t) = \frac{mbce^{-ct}}{1 - be^{-ct}} f_R(t) \quad \text{and} \quad f_R^{(2)}(t) = -c \frac{1 - mbe^{-ct}}{1 - be^{-ct}} f_R^{(1)}(t).$$

From (2.2) and (3.1), we can write

$$F_R^{(2)}(t) = [((2p - cq) - cpt) + ((mbcq - 2bp) + bcmpt)e^{-ct}] \frac{f_R^{(1)}(t)}{1 - be^{-ct}}.$$

It follows that

$$Q(t) = ((2p - cq) - cpt) + ((mbcq - 2bp) + bcmpt)e^{-ct}$$

and

$$g(t) = \frac{1}{1 - be^{-ct}}. \tag{3.3}$$

Two sets of conditions on the parameters are allowed here to ensure the increasing character of Richards function with the existence of an inflection point.

First case: Richards⁽¹⁾ with $a > 0$, $b > 0$, $c > 0$, $m > 1$, and $T_0 = \frac{\ln(b)}{c}$.

Second case: Richards⁽²⁾ with $a > 0$, $b < 0$, $c > 0$, $m < 0$, and $T_0 = -\infty$.

Then, Theorem 2 with $B = cp$, $D = cmbp > 0$ and $\gamma = c > 0$ ensures in both cases the existence of a unique $T^* > t^*$ such that the condition (3.2) holds.

3.3 Gompertz model $f_G(t) = e^{-be^{-ct}}$

It is well known that the Gompertz model is a limiting case of Richards⁽¹⁾ model for $m_R \rightarrow +\infty$, $b_R \rightarrow 0^+$, and $m_R b_R \rightarrow b_G$, and of Richards⁽²⁾ model for $m_R \rightarrow -\infty$, $b_R \rightarrow 0^-$, and $m_R b_R \rightarrow b_G$. This function is well defined on \mathbb{R} and $T_0 = -\infty$. It is a strictly increasing function with one inflection point for $b > 0$ and $c > 0$. We have

$$f_G^{(1)}(t) = bce^{-ct} f_G(t) \quad \text{and} \quad f_G^{(2)}(t) = -c[1 - be^{-ct}] f_G^{(1)}(t).$$

Hence, from (2.2) and (3.1), we get

$$F_G^{(2)}(t) = [((2p - cq) - cpt) + (bcq + bcpt)e^{-ct}] f_G^{(1)}(t),$$

then

$$Q(t) = ((2p - cq) - cpt) + (bcq + bcpt)e^{-ct},$$

and $g(t) = 1$. Thus, Theorem 2 with $B = cp$, $D = bcp$ and $\alpha = c > 0$ provides that there exists a unique $T^* > t^*$ such that the condition (3.2) holds.

3.4 Logistic model $f_L(t) = \frac{1}{1+be^{-ct}}$

Obviously, the Logistic model is a particular case of the Richards⁽²⁾ model with the parameter m equal to -1 and b is replaced by $-b$. It is a well defined function on \mathbb{R} and $T_0 = -\infty$. We must have $b > 0$ and $c > 0$ to have one inflection point at $t^* = \frac{\ln(b)}{c}$. Since

$$f_L^{(1)}(t) = \frac{bce^{-ct}}{1 + be^{-ct}} f_L(t) = cf_L(t)[1 - f_L(t)]$$

and

$$f_L^{(2)}(t) = -\frac{c(1 - be^{-ct})}{1 + be^{-ct}} f_L^{(1)}(t) = c[1 - 2f_L(t)]f_L^{(1)}(t),$$

Hence, from (2.2) and (3.1), it follows that

$$F_L^{(2)}(t) = [((2p - cq) - cpt) + (b(2p + cq + bcpt))e^{-ct}] \frac{f_L^{(1)}(t)}{1 + be^{-ct}}.$$

Hence we set

$$Q(t) = ((2p - cq) - cpt) + (b(2p + cq + bcpt))e^{-ct},$$

and

$$g(t) = \frac{1}{1 + be^{-ct}}.$$

From Theorem 2 with $B = cp$, $D = bcp$, and $\alpha = c > 0$ we have the existence of a unique $T^* > t^*$ such that the condition (3.2) holds.

3.5 Bridge model $f_B(t) = 1 - e^{-mt^b}$

The Bridge function is well defined when $t \geq T_0 = 0$. Moreover, we assume that $b \geq 1$ and $m > 0$ to have an increasing function with one inflection point. We have

$$f_B^{(1)}(t) = mbt^{b-1}e^{-mt^b} \quad \text{and} \quad f_B^{(2)}(t) = \frac{(b-1) - mbt^b}{t} f_B^{(1)}(t).$$

From (3.1) and (2.2), we obtain

$$F_B^{(2)}(t) = [(q(b-1) + (b+1)pt) - (mbq + mbpt)t^b] \frac{f_B^{(1)}(t)}{t}.$$

Then we have

$$Q(t) = (q(b-1) + (b+1)pt) - mb(q + pt)t^b, \quad g(t) = \frac{1}{t}.$$

Thus Theorem 1 with $B = (b+1)p > 0$, $D = mbp > 0$ and $\gamma = b > 1$ ensures the existence of a unique $T^* > t^*$ such that the condition (3.2) holds.

4 Numerical Examples

In this section we use the six models presented in the last section to fit some synthetic data sets $\{(t_k, y_k)\}_{k=1}^K$. Because the numerical behavior of the original models of Table 1 has been studied elsewhere, in this section we focus on the asymptote of the modified models of Table 2. Firstly, we compare the values of the two parameters of the asymptote, the slope p and the y -intercept q , computed from the fitting of the six models on the same data sets. Because a comparison of the models based on their best fits depends widely on the data set, we briefly mention the quality of the fitting, we compare and rank the fitting of the models using the value of the optimal criteria. Secondly, for a given model, the Richards⁽²⁾ model, we illustrate the effect of increasing the density of data to recover the values of the parameters p and q of the asymptote.

4.1 Numerical computation

The parameter vector (p, q, θ) of each model $F(t)$ is estimated by minimizing the least squares criteria

$$Z(p, q, \theta) = \frac{1}{2} \sum_{k=1}^K [F(t_k; (p, q, \theta)) - y_k]^2$$

over the feasible set of parameters Θ as defined for each model in the last two columns of Table 2. We used the MATLAB least squares function “lsqnonlin” to solve these problems. The MATLAB function “fsolve” was used to solve the equation $Q(t) = 0$ to obtain an approximation T^{**} of the abscissa T^* of the inflection point (T^*, F^*) for each modified model, and the approximate ordinate F^{**} is given by $F^{**} = F(T^{**})$. Numerical results of our simulations include the optimal solution (p^*, q^*, θ^*) , the optimal value of the criteria Z^* , its mean optimal value $Z^*/K = Z(p^*, q^*, \theta^*)/K$, and an approximation (T^{**}, F^{**}) of the inflection point (T^*, F^*) .

4.2 First example

4.2.1 Data sets

To generate the first series of data, we have first chosen the exponential model $F_E(t; p, q, \theta) = (pt + q)e^{-\alpha/t^\gamma}$, where $\theta = (\alpha, \gamma)$, because the shape of its graph looks like the graph of the models under study. We have used the value $(p, q, \theta) = (0.5, 100, 3000, 1.4)$ for its parameters. We have generated 10 random uniformly distributed abscissae on the interval $(0, 2000]$, by choosing $t_k \sim \mathcal{U}(0, 2000]$, and 10 random uniformly distributed abscissae on the $(2000, 4000]$, by choosing $t_k \sim \mathcal{U}(2000, 4000]$. Then the ordinates are computed by $y_k = F_E(t_k; p, q, \theta) + r_k$ for $k = 1, \dots, 20$, where $r_k \sim \mathcal{N}(0, \sigma^2)$ with $\sigma = 10$.

The six models of Table 2 were then fitted to the 20 data on $(0, 4000]$. With each of these optimal models we have generated 20 other data points, ten on each subintervals $(0, 2000]$ and $(2000, 4000]$, as we did before with the exponential model. We have added together these data points to get two sets

of 70 data points on each intervals $(0, 2000]$ and $(2000, 4000]$. In this way we can say that the data represent all the models.

We have considered the best fitting for the following situations:

- on the interval $(0, 2000]$ with 70 data points;
- on the interval $(0, 4000]$ with 140 data points;
- on the interval $(0, 4000]$ with 70 data points (chosen randomly in the preceding data set with 35 data points on each subinterval $(0, 2000]$ and $(2000, 4000]$);

The results are reported in Table 3. For each data set, and on each model F , we have tested 10 different starting points (p_0, q_0, θ_0) randomly chosen in the feasible set Θ using continuous uniform distributions. For each model and each data set, when the starting values are chosen around the values of the parameters used to generate the data, the algorithm has always converged to the same optimal solution from any one of these starting points.

4.2.2 Numerical results and discussion

From Table 3 we observe that the asymptotes recovered by the Gompertz, Logistic, Richards⁽¹⁾, Richards⁽²⁾, and Bridge models have the slope p^* and y-intercept q^* relatively close together. These models recover almost the same asymptote. For the Michaelis–Menten model the slope p^* is also close to the values obtained by the preceding 5 models, but the y-intercept q^* is slightly bigger. This differences in q^* decreases if we increase the length T of the interval, from 2000 to 4000, maintaining the same density K/T of data as shown when we compare the parameter values of Table 3.

Based on the optimal value of the criteria Z^*/K , the numerical results summarized in Table 4 indicates that the best fit is obtained for the Richards⁽²⁾ model followed by the Bridge model, but the differences between the Z^*/K for the 6 models are rather small. We also observe the same results for the Richards⁽¹⁾ and the Gompertz models because $m_R b_R \approx b_G$. Obviously, the rank for the models might change depending on the data set under consideration. We have included, in Figures 2 and 3, the graphs of the best model, the Richards⁽²⁾ model, and the graphs of the worst one, the Michaelis–Menten model, on the intervals $(0, 2000]$ and $(0, 4000]$ to illustrate that globally the fitting is good for all models. We can see slight differences on the graphs of the asymptotes. Let us observe that when we increase the length T with the same data density, the mean value Z^*/K decreases, which means we get more information for the fitting and obtain a more reliable asymptote when T is large.

4.3 Second example

4.3.1 Data sets

In this section we have tested Richards⁽²⁾ on different data sets $\{(t_k, y_k)\}_{k=1}^K$ with different values of K on the interval $(0, 4000]$. The data points $\{(t_k, y_k)\}_{k=1}^K$

Table 3. Optimal parameters, optimal value and an approximation of the inflection point (T^{**}, F^{**}) for each data set (K, T).

Michealis-Menten model: $F_M(t; p, q, \omega_0, k, c)$								
K	T	p^*	q^*	ω_0^*	k^*	c^*	Z^*	(T^{**}, F^{**})
70	2000	0.4933	142.04	0.09506	403.34	2.9780	3840.4	(420.78, 201.36)
140	4000	0.5102	106.45	0.15396	393.03	3.4883	6820.6	(415.07, 196.38)
70	4000	0.5117	102.53	0.16045	398.77	3.5697	3203.3	(421.75, 198.02)
T	K	p^*	q^*	b^*	c^*	m^*	Z^*	(T^{**}, F^{**})
Richards ⁽¹⁾ model: $F_R(t; p, q, b, c, m)$								
70	2000	0.5090	103.28	0.00038	0.00536	11379	3752.5	(436.83, 214.60)
140	4000	0.5128	96.957	0.00047	0.00551	9447.4	6587.9	(430.23, 209.89)
70	4000	0.5136	95.341	0.00036	0.00540	12032	3115.8	(438.99, 213.49)
Richards ⁽²⁾ model: $F_R(t; p, q, b, c, m)$								
70	2000	0.5155	91.963	-3.1315	0.00631	-2.1955	3709.9	(449.43, 223.57)
140	4000	0.5137	94.305	-2.8042	0.00622	-2.3696	6470.6	(449.68, 223.76)
70	4000	0.5148	91.792	-4.3383	0.00639	-1.7672	3068.2	(462.23, 229.84)
Gompertz model: $F_G(t; p, q, b, c)$								
70	2000	0.5090	103.28	4.3372	0.00536		3752.4	(436.83, 214.60)
140	4000	0.5128	96.956	4.4386	0.00551		6587.8	(430.24, 209.89)
70	4000	0.5136	95.341	4.3621	0.00540		3115.8	(438.99, 213.49)
Logistic model: $F_L(t; p, q, b, c)$								
70	2000	0.5206	80.347	11.916	0.00740		3777.8	(460.61, 236.87)
140	4000	0.5146	91.473	11.711	0.00721		6603.4	(468.52, 237.67)
70	4000	0.5155	89.611	11.964	0.00716		3110.1	(474.90, 239.16)
Bridge model: $F_B(t; p, q, b, m)$								
70	2000	0.5189	87.746	1.7673		$2.473e-5$	3744.3	(452.40, 234.05)
140	4000	0.5144	92.235	1.7492		$2.679e-5$	6528.3	(467.86, 237.98)
70	4000	0.5151	90.758	1.7476		$2.642e-5$	3096.6	(476.07, 240.97)

Table 4. Ranking criterion values for the models.

Model	Data set $K = 70$ and $T = 2000$			Data set $K = 140$ and $T = 4000$		
	Z^*	Z^*/K	Rank	Z^*	Z^*/K	Rank
Michaelis–Menten	3840.4	54.863	6	6820.6	48.719	6
Richards ⁽¹⁾	3752.4	53.606	3	6587.8	47.056	3
Richards ⁽²⁾	3709.9	52.999	1	6470.6	46.219	1
Gompertz	3752.4	53.606	3	6587.8	47.056	3
Logistic	3777.8	53.969	5	6603.4	47.167	5
Bridge	3744.3	53.490	2	6528.3	46.631	2

are generated, for $k = 1, \dots, K$, by

$$y_k = (pt_k + q)F_R(t_k; \theta) + r_k, \tag{4.1}$$

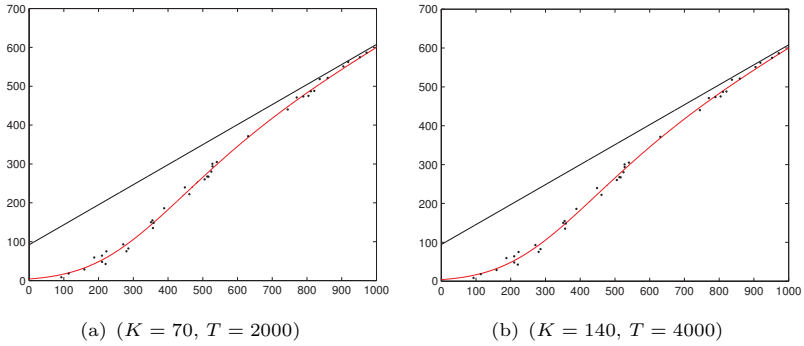


Figure 2. Graphic representation of the synthetic data points $\{t_k, y_k\}_{k=1}^K$ with the modified Richards⁽²⁾ model $F_R(t; p, q, \theta) = (pt + q)f_R(t; \theta)$ and its asymptote at the optimal solution (p^*, q^*, θ^*) .

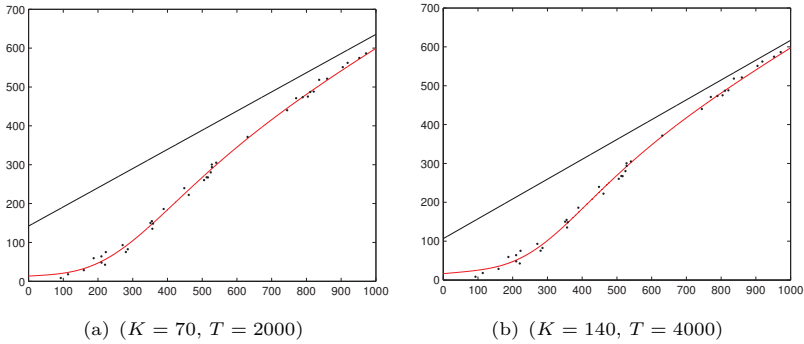


Figure 3. Graphic representation of the synthetic data points $\{t_k, y_k\}_{k=1}^K$ with the modified Michaelis–Menten model $F_M(t; p, q, \theta) = (pt + q)f_M(t; \theta)$ and its asymptote $m(t) = pt + q$ at the optimal solution (p^*, q^*, θ^*) .

where $(p, q, \theta) = (0.5, 100, -3, 0.005, -2)$, $t_k \sim \mathcal{U}(0, 4000)$ is a uniformly distributed random variable over $(0, T]$, $r_k \sim \mathcal{N}(0, \sigma^2)$ is a random normal variable with mean $\mu = 0$ and standard deviation $\sigma = 4$. Numerical results are summarized in Table 5.

4.3.2 Numerical results and discussion

The results of the Table 5 indicate a monotonic convergence of the estimated values of the parameters through the true values of the parameters as the density K of the data increases in the interval.

5 Conclusion

In this paper we have extended a class of standard growth curves with horizontal asymptote in order to obtain growth curves with linear asymptote. The

Table 5. Richards⁽²⁾ model ($T = 4000$): Optimal parameters, optimal value and an approximation of the inflection point (T^{**}, F^{**}) for different data sets with an increasing degree of freedom.

Richards ⁽²⁾ model: $F_R(t; p, q, b, c, m)$								
True values	p	q	b	c	m	(T^*, F^{**})		
	0.5	100	-3	0.005	-2	(555.9, 268.63)		
K	p^*	q^*	b^*	c^*	m^*	Z^*	Z^*/K	(T^*, F^{**})
20	0.5055	91.41	-10.08	0.00574	-0.7447	103.38	5.169	(594.6, 300.33)
40	0.5053	92.52	-6.706	0.00569	-0.9774	306.57	7.664	(585.9, 287.99)
60	0.5048	93.27	-6.951	0.00566	-1.1920	388.41	6.474	(571.1, 280.08)
80	0.5047	93.31	-5.708	0.00557	-1.2013	452.86	5.661	(573.9, 278.83)
100	0.5021	94.73	-4.828	0.00543	-1.3373	715.41	7.154	(561.8, 270.79)
120	0.5013	96.69	-4.688	0.00538	-1.3561	772.29	6.436	(561.3, 272.61)
140	0.5007	98.16	-3.774	0.00538	-1.5728	921.59	6.598	(558.5, 266.42)
160	0.5007	98.54	-3.661	0.00507	-1.7069	1139.2	7.120	(558.8, 266.47)
180	0.5005	98.89	-3.177	0.00507	-1.9521	1362.5	7.569	(559.6, 271.94)
200	0.5000	99.89	-2.924	0.00495	-1.9814	1497.7	7.489	(557.3, 270.24)

objective of the present study was to present such models and show their performances on synthetic data sets. In addition to their increasing character with an oblique asymptote, the resulting models provide curves with a single inflection point. Other extensions of these models to obtain curvilinear asymptote and growth curves with several inflection points could be of interest. These extensions and their applications will be presented in a future paper.

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