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Mellin Transform of Dirichlet L-Functions with Primitive Character

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Abstract. In the paper, meromorphic continuation for the modified Mellin transform of Dirichlet L-functions with primitive character is obtained.

Keywords: Dirichlet L-function, meromorphic continuation, modified Mellin transform.

AMS Subject Classification: 11M06; 44A15.

1 Introduction

In the theory of zeta and L-functions, the moments play an important role. For the investigation of the moments of the Riemann zeta-function $\zeta(s)$, $s=\sigma+it$, on the critical line

$$I_k(T) \stackrel{\text{def}}{=} \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} \mathrm{d}t, \quad k \geq 0,\ T \to \infty,$$

Y. Motohashi in [19] and [20] introduced and applied the modified Mellin transforms

$$\mathcal{Z}_k(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2}+ix\right)\right|^{2k} x^{-s} \mathrm{d}x, \quad \sigma > 1.$$

In a series of works [6, 7, 8, 9, 10, 11, 13, 14, 18], see also [15, 16, 17], the theory of the transforms $\mathcal{Z}_k(s)$ was developed, and gave important results for the moments $I_k(T)$.

The method of modified Mellin transforms also can be applied for investigation of moments of the Dirichlet L-functions $L(s,\chi)$. For this, analytic theory of modified Mellin transforms of these functions is needed. In [14], the modified

Mellin transforms

$$\mathcal{Z}_k(s,L) \stackrel{\text{def}}{=} \sum_{\chi \bmod q} \int_1^\infty \left| L\left(\frac{1}{2}+ix,\chi\right)\right|^{2k} x^{-s}\mathrm{d}x$$

were considered.

In this paper, we study the modified Mellin transform of individual Dirichlet L-functions $L(s,\chi)$ with primitive character χ

$$\mathcal{Z}_1(s,\chi) \stackrel{\text{def}}{=} \int_1^\infty \left| L\left(\frac{1}{2}+ix,\chi\right)\right|^{2} x^{-s}\mathrm{d}x, \quad \sigma>1.$$

The aim of this paper is to obtain a meromorphic continuation of $\mathcal{Z}_1(s,\chi)$ to the whole complex plane. In [2], the meromorphic continuation for the function $\mathcal{Z}_1(s,\chi_0)$, where χ_0 is the principal character modulo q, has been obtained.

Define

$$b = \begin{cases} 0 & \text{if } \chi(-1)=1, \\ 1 & \text{if } \chi(-1)=-1 \end{cases}, \qquad c(q)=\sum_{a=1}^{q} \overline{\chi}(a)(q,a-1),$$

where $\overline{\chi}(a)$ is a conjugate Dirichlet character modulo q, and $(q,a-1)$ denotes the greatest common divisor. Let, as usual, γ_0 denote Euler's constant, and B_j stand for the j-th Bernoulli number. Then the following theorem is true.

Theorem 1. *The function $\mathcal{Z}_1(s,\chi)$ has a meromorphic continuation to the whole complex plane.*

1. If $c(q)\neq 0$, then it has a double pole at the point $s=1$, and the main part of its Laurent expansion at this point is

$$\mathcal{Z}_1(s,\chi)=\frac{i^b}{q}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\left(\frac{1}{(s-1)^2}+\frac{2\gamma_0+\log{(q,a-1)^2/2\pi q}}{s-1}\right)+\cdots.$$

The other poles of $\mathcal{Z}_1(s,\chi)$ are the simple poles at the points $s=-(2j-1)$, $j\in\mathbb{N}$, and

$$\operatorname*{Res}_{\substack{s=-(2j-1)\\ j\in\mathbb{N}}} \mathcal{Z}_1(s,\chi)=\frac{i^{b-2j}(1-2^{1-2j})B_{2j}}{2jq}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1).$$

2. If $c(q)=0$, then $\mathcal{Z}_1(s,\chi)$ is an entire function.

2 Connection Between the Laplace and Mellin Transforms

The proof of Theorem 1 is based on the formula for the Laplace transform $\mathfrak{L}(s,\chi)$ defined by

$$\mathfrak{L}(s,\chi) \stackrel{\text{def}}{=} \int_0^\infty \left| L\left(\frac{1}{2}+ix,\chi\right)\right|^{2} e^{-sx}\mathrm{d}x.$$

This formula have been obtained in [3], and we state it as a separate lemma. Let

$$d(m) = \sum_{d|m} 1$$

be the divisor function, let

$$G(\chi) = \sum_{l=1}^{q} \chi(l) e^{2\pi i l/q}$$

denote the Gauss sum,

$$\epsilon(\chi) = \frac{G(\chi)}{\sqrt{q}}, \qquad \epsilon_1(\chi) = -\frac{G(\chi)}{\sqrt{q}}, \qquad E(\chi) = \begin{cases} \epsilon(\chi) & \text{if } b = 0, \\ \epsilon_1(\chi) & \text{if } b = 1. \end{cases}$$

Lemma 1. *Let $\{s \in \mathbb{C} : 0 < \sigma < \pi\}$, and let χ be a primitive character mod $q > 1$. Then*

$$\mathfrak{L}(s,\chi) = \frac{2\pi i^b e^{-\frac{is}{2}}}{\sqrt{q}E(\chi)} \sum_{m=1}^{\infty} d(m)\chi(m) \exp\left\{-\frac{2\pi i m}{q} e^{-is}\right\} + \lambda(s,\chi),$$

where the function $\lambda(s,\chi)$ is analytic in the strip $\{s \in \mathbb{C} : |\sigma| < \pi\}$, and, for $|\sigma| \le \theta$, $0 < \theta < \pi$, the estimate

$$\lambda(s,\chi) = O\big((1+|s|)\big)^{-1}$$

is valid.

Now, from the definitions of $\mathcal{Z}_1(s,\chi)$, $\mathfrak{L}(w,\chi)$, $\Gamma(s)$, we have

$$\mathcal{Z}_1(s,\chi) = \frac{1}{\Gamma(s)} \int_0^{\infty} \mathfrak{L}(w,\chi) w^{s-1} \mathrm{d}w. \tag{2.1}$$

In formula (2.1), we change the integration over the positive real axis. In the same way as in [2], for $0 \le \alpha < \frac{\pi}{2}$, using the residue theorem, we get

$$\mathcal{Z}_1(s,\chi) = \frac{1}{\Gamma(s)} \int_0^{\infty e^{i\alpha}} \mathfrak{L}(w,\chi) w^{s-1} \mathrm{d}w.$$

Fixing the point $w_0 = |w_0|e^{i\alpha}$ with $0 < \operatorname{Re} w_0 < \pi$, and using Lemma 1, we define the functions

$$\begin{aligned}
\mathcal{Z}_{11}(s,\chi) &= \frac{1}{\Gamma(s)} \int_0^{w_0} \lambda(w,\chi) w^{s-1} \mathrm{d}w, \\
\mathcal{Z}_{12}(s,\chi) &= \frac{1}{\Gamma(s)} \int_{w_0}^{\infty e^{i\alpha}} \left(\int_1^{\infty} \left| L\left(\frac{1}{2} + ix, \chi\right) \right|^2 e^{-wx} \mathrm{d}x \right) w^{s-1} \mathrm{d}w, \\
\mathcal{Z}_{13}(s,\chi) &= \frac{2\pi i^b}{\Gamma(s)\sqrt{q}E(\chi)} \sum_{k=1}^{\infty} d(k)\chi(k) \int_0^{w_0} e^{-\frac{iw}{2}} \exp\left\{-\frac{2\pi i k}{q} e^{-iw}\right\} w^{s-1} \mathrm{d}w,
\end{aligned}$$

$$\mathcal{Z}_1(s,\chi)=\sum_{j=1}^{3}\mathcal{Z}_{1j}(s,\chi). \tag{2.2}$$

By partial integration, we find that the functions $\mathcal{Z}_{11}(s,\chi)$ and $\mathcal{Z}_{12}(s,\chi)$ are entire, thus, it remains to study the function $\mathcal{Z}_{13}(s,\chi)$.

3 Auxiliary Results

For the investigation of the function $\mathcal{Z}_{13}(s,\chi)$, some auxiliary results are needed. We state these results as separate lemmas. The obtained formulae involve the Estermann zeta-function.

For $\alpha\in\mathbb{C}$, let

$$\sigma_\alpha(m)=\sum_{d|m}d^\alpha$$

be the generalized divisor function. Let $l\geq 1$ and $(k,l)=1$. The Estermann zeta-function $E\left(s;\frac{k}{l},\alpha\right)$, for $\sigma>\max(1+\operatorname{Re}\alpha,1)$, is defined by the series

$$E\left(s;\frac{k}{l},\alpha\right)=\sum_{m=1}^{\infty}\frac{\sigma_\alpha(m)}{m^s}\exp\left\{2\pi im\frac{k}{l}\right\}.$$

The function $E\left(s;\frac{k}{l},\alpha\right)$, for $\alpha=0$, was introduced by T. Estermann in [5]. It is known, see, for example, [12], that $E\left(s;\frac{k}{l},0\right)$ has the Laurent series expansion

$$E\left(s;\frac{k}{l},0\right)=\frac{1}{l}\left(\frac{1}{(s-1)^2}+\frac{2\gamma_0-2\log l}{s-1}+c_0+c_1(s-1)+\cdots\right). \tag{3.1}$$

Let

$$\delta=\begin{cases}1 & \text{if } \operatorname{Im}z>0,\\ -1 & \text{if } \operatorname{Im}z<0,\end{cases}$$

k and l be coprime integers, $z\in\mathbb{C}\backslash\{0\}$, and let $\overline{k}$ and k be related by the congruence $k\overline{k}\equiv 1(\operatorname{mod} l)$. Moreover, denote by a_0^+ and a_0^- the constant terms in (3.1) for $E(s;\frac{\overline{k}}{l},0)$ and $E(s;-\frac{\overline{k}}{l},0)$, respectively. Define

$$\Phi\left(z;\frac{k}{l}\right)=\sum_{m=1}^{\infty}d(m)e^{2\pi i\frac{km}{l}}e^{-mz}-\frac{\gamma_0-2\log l-\log z}{lz}$$

and, for $1<b<2$,

$$I(z,b)=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\left(\frac{2\pi}{l}\right)^{1-2w}\Gamma(w)\bigg(\big(\sin(\pi w)\big)^{-1}E\left(w;\frac{\overline{k}}{l},0\right)$$
$$+\big(\cot(\pi w)+\delta i\big)E\left(w;-\frac{\overline{k}}{l},0\right)\bigg)z^{1-w}\mathrm{d}w.$$

Then, in [1], the following transformation formula for $\Phi(z^{-1};\frac{k}{l})$ has been proved.

Lemma 2. *If* $\operatorname{Re} z > 0$ *and* $\operatorname{Im} z \neq 0$, *then for the function* $\Phi(z; \frac{k}{l})$ *the transformation formula*

$$\Phi\left(z^{-1}; \frac{k}{l}\right) = -\frac{2\pi i \delta z}{l} \sum_{m=1}^{\infty} d(m) e^{-2\pi i m \frac{\overline{k}}{l}} e^{-\frac{4\pi^2 m z}{l^2}} + \frac{l}{2\pi^2}\left(a_0^{+} - a_0^{-}\right) + \frac{1}{4} + I(z,b)$$

is valid.

In the future, we will express the exponential sum $\sum_{m=1}^{\infty} d(m)\chi(m) \times \exp\{\frac{2\pi i m}{q}\} m^{-s}$ by the Gaussian sum. For this, we will use the following lemma.

Lemma 3. *For* $\sigma > 1$,

$$\sum_{m=1}^{\infty} d(m)\chi(m) \exp\left\{-\frac{2\pi i m}{q}\right\} m^{-s} = \frac{1}{G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) E\left(s; \frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}}, 0\right).$$

Proof. It is well known, see, for example, [4], that, for every $m \in \mathbb{N}$,

$$\chi(m) G(\overline{\chi}) = \sum_{a=1}^{q} \chi(a) e^{2\pi i m a/q}.$$

Using this, we get

$$\begin{aligned} G(\overline{\chi}) & \sum_{m=1}^{\infty} d(m)\chi(m) \exp\left\{-\frac{2\pi i m}{q}\right\} m^{-s} \\ &= \sum_{m=1}^{\infty} d(m) \exp\left\{-\frac{2\pi i m}{q}\right\} m^{-s} \sum_{a=1}^{q} \overline{\chi}(a) e^{2\pi i m a/q} \\ &= \sum_{a=1}^{q} \overline{\chi}(a) \sum_{m=1}^{\infty} d(m) e^{2\pi i m (a-1)/q} m^{-s} = \sum_{a=1}^{q} \overline{\chi}(a) E\left(s; \frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}}, 0\right), \end{aligned}$$

and the claim of the lemma follows.

Lemma 4. *Suppose that* $a > 0$ *and* $b > 0$. *Then*

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(s) b^{-s} \mathrm{d}s = e^{-b}.$$

The lemma is the well-known Mellin formula, see, for example, [21].

Lemma 5. *Let* $0 < a < 1$. *Then*

$$\Phi\left(z; \frac{k}{l}\right) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(w) E\left(w; \frac{k}{l}, 0\right) z^{-w} \mathrm{d}w.$$

Proof. For $\operatorname{Re} z > 0$, the series

$$\sum_{m=1}^{\infty} d(m) \exp\left\{2\pi i m \frac{k}{l}\right\} e^{-mz}$$

converges absolutely, therefore, by Lemma 4 and definition of the Estermann zeta-function, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty} d(m) \exp\left\{2\pi i m \frac{k}{l}\right\} e^{-mz} \\
&\quad = \sum_{m=1}^{\infty} d(m) \exp\left\{2\pi i m \frac{k}{l}\right\} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(w)(mz)^{-w} \mathrm{d}w \\
&\quad = \frac{1}{2\pi i} \int_{2-\infty}^{2+\infty} \Gamma(w) E\left(w; \frac{k}{l}, 0\right) z^{-w} \mathrm{d}w. \qquad (3.2)
\end{aligned}$$

Now we move the line of integration in (3.2) to the left. Let $0 < a < 1$. From formula (3.1), we see that the function $E\left(w; \frac{k}{l}, 0\right)$ has a double pole at the point $w = 1$, therefore, by the residue theorem,

$$\begin{aligned}
\sum_{m=1}^{\infty} d(m) \exp\left\{2\pi i m \frac{k}{l}\right\} e^{-mz} &= \frac{1}{2\pi i} \int_{a-\infty}^{a+\infty} \Gamma(w) E\left(w; \frac{k}{l}, 0\right) z^{-w} \mathrm{d}w \\
&\quad + \operatorname*{Res}_{w=1} \Gamma(w) E\left(w; \frac{k}{l}, 0\right) z^{-w}.
\end{aligned}$$

Clearly,

$$\Gamma(w) = 1 - \gamma_0(w-1) + \frac{\Gamma''(1)(w-1)^2}{2} + \cdots$$

and

$$z^{-w} = z^{-1} e^{-(w-1)\log z} = z^{-1}\left(1 - (w-1)\log z + \frac{(w-1)^2 \log^2 z}{2} + \cdots\right).$$

Hence,

$$\operatorname*{Res}_{w=1} \Gamma(w) E\left(w; \frac{k}{l}, 0\right) z^{-w} = \frac{\gamma_0 - 2\log l - \log z}{lz},$$

and the assertion of the lemma follows.

4 Entire Parts of $\mathcal{Z}_{13}(s,\chi)$

In the definition of the function $\mathcal{Z}_{13}(s,\chi)$, we take $e^{-iw} = 1 + \frac{1}{z}$, and let $z_0 = (e^{-iw_0} - 1)^{-1}$. This leads to the formula

$$\begin{aligned}
\mathcal{Z}_{13}(s,\chi) &= \frac{2\pi i^{b+s}}{\Gamma(s)\sqrt{q}E(\chi)} \sum_{k=1}^{\infty} d(k)\chi(k) e^{-\frac{2\pi i k}{q}} \\
&\quad \times \int_{z_0}^{\infty} z^{-2}\left(1 + \frac{1}{z}\right)^{-\frac{1}{2}} \left(\log\left(1 + \frac{1}{z}\right)\right)^{s-1} e^{-\frac{2\pi i k}{qz}} \mathrm{d}z, \qquad (4.1)
\end{aligned}$$

where the integrals are taken over the curve $z=(e^{-ire^{i\alpha}}-1)^{-1}$.

For $|z|>1$, from Lemmas 2–5, taking $\frac{qz}{2\pi i}$ in place of z, and having in mind that, in this case $\mathrm{Im}(\frac{qz}{2\pi i})<0$, therefore, $\delta=-1$, we find

$$\begin{aligned}
&\sum_{k=1}^{\infty} d(k)\chi(k)e^{-\frac{2\pi ik}{q}}e^{-\frac{2\pi ik}{qz}}\\
&\quad=\frac{1}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\varPhi\left(\frac{2\pi i}{qz};\frac{\frac{a-1}{(q,a-1)}}{\frac{q}{(q,a-1)}}\right)+\frac{(\gamma_0-\log\frac{2\pi iq}{(q,a-1)^2z})z(q,a-1)}{2\pi i}\\
&\quad=\frac{1}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\varPhi\left((\frac{qz}{2\pi i})^{-1};\frac{\frac{a-1}{(q,a-1)}}{\frac{q}{(q,a-1)}}\right)+\frac{(\gamma_0-\log\frac{2\pi iq}{(q,a-1)^2z})z(q,a-1)}{2\pi i}\\
&\quad=\frac{1}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\left(\frac{2\pi i(q,a-1)}{q}\cdot\frac{qz}{2\pi i}\right)\sum_{k=1}^{\infty}d(k)e^{-2\pi ik\overline{\frac{(a-1)/(q,a-1)}{q/(q,a-1)}}}\\
&\qquad\times e^{-\frac{4\pi^2k(q,a-1)^2\cdot\frac{qz}{2\pi i}}{q^2}}+\frac{q}{2\pi^2(q,a-1)}(a_{0a}^{+}-a_{0a}^{-})+\frac{1}{4}+I\left(\frac{qz}{2\pi i},b\right)\\
&\qquad+\frac{(\gamma_0-\log\frac{2\pi iq}{(q,a-1)^2z})z(q,a-1)}{2\pi i}\\
&\quad=\frac{z}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\sum_{k=1}^{\infty}d(k)e^{-2\pi ik\overline{\frac{(a-1)/(q,a-1)}{q/(q,a-1)}}}e^{\frac{2\pi ik(q,a-1)^2z}{q}}\\
&\qquad+\frac{q}{2\pi^2(q,a-1)}(a_{0a}^{+}-a_{0a}^{-})+\frac{(\gamma_0-\log\frac{2\pi iq}{(q,a-1)^2z})z(q,a-1)}{2\pi i}.
\end{aligned}\tag{4.2}$$

Now if $|z|<1$, we take $\frac{qi}{2\pi z(q,a-1)^2}$ in place of z. Since $\mathrm{Im}(\frac{qi}{2\pi z(q,a-1)^2})>0$, we have that $\delta=1$, therefore, from the mentioned lemmas, it follows that

$$\begin{aligned}
&\sum_{k=1}^{\infty} d(k)\chi(k)e^{-\frac{2\pi ik}{q}}e^{-\frac{2\pi ik}{qz}}\\
&\quad=\frac{1}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\varPhi\left(\frac{2\pi i}{qz};\frac{\frac{a-1}{(q,a-1)}}{\frac{q}{(q,a-1)}}\right)+\frac{(\gamma_0-\log\frac{2\pi iq}{(q,a-1)^2z})z(q,a-1)}{2\pi i}\\
&\quad=\frac{1}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\varPhi\left(\left(\frac{4\pi^2(q,a-1)^2}{q^2}\cdot\frac{qi}{2\pi z(q,a-1)^2}\right);\frac{\frac{a-1}{(q,a-1)}}{\frac{q}{(q,a-1)}}\right)\\
&\qquad+\frac{(\gamma_0-\log\frac{2\pi iq}{(q,a-1)^2z})z(q,a-1)}{2\pi i}\\
&\quad=\frac{1}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\left\{-\frac{q}{2\pi i(q,a-1)\frac{qi}{2\pi z(q,a-1)^2}}\varPhi\left(\frac{2\pi z(q,a-1)^2}{qi};-\overline{\frac{(\frac{(a-1)}{(q,a-1)})}{\frac{q}{(q,a-1)}}}\right)\right.\\
&\qquad+\frac{q^2}{4\pi^3i(q,a-1)^2\cdot\frac{qi}{2\pi z(q,a-1)^2}}(a_{0a}^{+}-a_{0a}^{-})+\frac{q}{8\pi i(q,a-1)\cdot\frac{qi}{2\pi z(q,a-1)^2}}\\
&\qquad+\frac{q}{2\pi i(q,a-1)\cdot\frac{qi}{2\pi z(q,a-1)^2}}I\left(\frac{qi}{2\pi z(q,a-1)^2},b\right)
\end{aligned}$$

$$- \frac{(\gamma_0 - \log \frac{4\pi^2 qi}{2\pi z(q,a-1)^2})q}{4\pi^2(q,a-1)\cdot \frac{qi}{2\pi z(q,a-1)^2}} + \frac{(\gamma_0 - \log \frac{2\pi iq}{(q,a-1)^2 z})z(q,a-1)}{2\pi i}\Bigg\}$$

$$= \frac{z}{G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)\Bigg((q,a-1)\sum_{k=1}^{\infty} d(k) e^{-2\pi ik \overline{\frac{(a-1)/(q,a-1)}{q/(q,a-1)}}} e^{\frac{2\pi ik(q,a-1)^2 z}{q}}$$

$$- z(q,a-1)\frac{(\gamma_0 - \log(\frac{q^2}{(q,a-1)^2}\cdot\frac{2\pi z(q,a-1)^2}{qi}))}{\frac{q}{(q,a-1)}\cdot\frac{2\pi z(q,a-1)^2}{qi}} - \frac{qz}{2\pi^2}(a_{0a}^{+} - a_{0a}^{-})$$

$$- \frac{z(q,a-1)}{4} - z(q,a-1) I\left(\frac{qi}{2\pi z(q,a-1)^2}, b\right)\Bigg)$$

$$= \frac{z}{G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)\Bigg((q,a-1)\sum_{k=1}^{\infty} d(k) e^{-2\pi ik \overline{\frac{(a-1)/(q,a-1)}{q/(q,a-1)}}} e^{\frac{2\pi ik(q,a-1)^2 z}{q}}$$

$$- \frac{qz}{2\pi^2}(a_{0a}^{+} - a_{0a}^{-}) - \frac{z(q,a-1)}{4} - z(q,a-1) I\left(\frac{qi}{2\pi z(q,a-1)^2}, b\right)\Bigg). \tag{4.3}$$

We denote by $\hat{z}$ the point on the path of integration in formula (4.1) such that $|\hat{z}| = 1$. Then, by (4.2) and (4.3), we write $\mathcal{Z}_{13}(s,\chi)$ in the form

$$\mathcal{Z}_{13}(s,\chi) = \sum_{j=1}^{6} I_{j3}(s,\chi),$$

where

$$I_{13}(s,\chi) = \frac{2\pi i^{b+s}}{\Gamma(s)\sqrt{q}E(\chi)G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)(q,a-1)\sum_{k=1}^{\infty} d(k) e^{-2\pi ik \overline{\frac{(a-1)/(q,a-1)}{q/(q,a-1)}}}$$
$$\times \int_{z_0}^{\infty} z^{-1}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}}\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} e^{\frac{2\pi ik(q,a-1)^2 z}{q}}\,\mathrm{d}z,$$

$$I_{23}(s,\chi) = -\frac{\sqrt{q}i^{b+s}}{\pi\Gamma(s)E(\chi)G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)(a_{0a}^{+} - a_{0a}^{-})$$
$$\times \int_{z_0}^{\hat{z}} z^{-1}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}}\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1}\mathrm{d}z,$$

$$I_{33}(s,\chi) = -\frac{2\pi i^{b+s}}{4\Gamma(s)\sqrt{q}E(\chi)G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)(q,a-1)$$
$$\times \int_{z_0}^{\hat{z}} z^{-1}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}}\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1}\mathrm{d}z,$$

$$I_{43}(s,\chi) = -\frac{2\pi i^{b+s}}{\Gamma(s)\sqrt{q}E(\chi)G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)(q,a-1)$$
$$\times \int_{z_0}^{\hat{z}} z^{-1}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}} I\left(\frac{qi}{2\pi z(q,a-1)^2}, b\right)\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1}\mathrm{d}z,$$

$$I_{53}(s,\chi) = \frac{\sqrt{q}i^{b+s}}{\pi\Gamma(s)E(\chi)G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\frac{(a_{0a}^{+}-a_{0a}^{-})}{(q,a-1)}$$
$$\times\int_{\hat{z}}^{\infty} z^{-2}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}}\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1}\mathrm{d}z$$

and

$$I_{63}(s,\chi) = \frac{i^{b+s-1}}{\Gamma(s)\sqrt{q}E(\chi)G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)$$
$$\times\int_{\hat{z}}^{\infty} z^{-1}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}}\left(\gamma_0-\log\frac{2\pi iq}{(q,a-1)^2z}\right)\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1}\mathrm{d}z.$$

All these parts are holomorphic, except for $I_{63}(s,\chi)$, which can produce the poles of $\mathcal{Z}_{13}(s,\chi)$.

5 Proof of Theorem 1

Using the Taylor series expansion, we find that

$$\begin{aligned}
&\left(1+\frac{1}{z}\right)^{-\frac{1}{2}}\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1}\\
&\quad=\left(1-\frac{1}{2z}+\frac{3}{4\cdot 2!z^2}-\dots\right)\cdot\left(\frac{1}{z}-\frac{1}{2z^2}+\frac{1}{3z^3}-\dots\right)^{s-1}\\
&\quad=z^{-s+1}\left(1-\frac{s}{2z}+b_2(s)\frac{1}{z^2}+\cdots\right).
\end{aligned}\tag{5.1}$$

This gives a new form of the function $I_{63}(s,\chi)$,

$$\begin{aligned}
I_{63}(s,\chi) &= \frac{i^{b+s-1}}{\Gamma(s)\sqrt{q}E(\chi)G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\sum_{k=0}^{\infty}b_k(s)\\
&\quad\times\int_{\hat{z}}^{\infty} z^{-s-k}\left(\gamma_0-\log\frac{2\pi iq}{(q,a-1)^2z}\right)\mathrm{d}z.
\end{aligned}\tag{5.2}$$

In virtue of

$$\int_{\hat{z}}^{\infty} z^{-s-k}\log z\,\mathrm{d}z = \frac{1}{s+k-1}\hat{z}^{-s-k+1}\log\hat{z}+\frac{\hat{z}^{-s-k+1}}{(s+k-1)^2},\tag{5.3}$$

for the term with $k=0$, we obtain

$$\begin{aligned}
&\frac{i^{b+s-1}}{\Gamma(s)\sqrt{q}E(\chi)G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\int_{\hat{z}}^{\infty} z^{-s}\left(\gamma_0-\log\frac{2\pi iq}{(q,a-1)^2z}\right)\mathrm{d}z\\
&\quad=\frac{i^{b+s-1}}{\Gamma(s)\sqrt{q}E(\chi)G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\\
&\qquad\times\left(\frac{\hat{z}^{-s+1}}{(s-1)^2}+\frac{(\gamma_0-\log\frac{2\pi iq}{(q,a-1)^2\hat{z}})\hat{z}^{-s+1}}{s-1}\right).
\end{aligned}\tag{5.4}$$

Thus, if $c(q) \neq 0$, the Mellin transform $\mathcal{Z}_k(s,\chi)$ has a double pole at $s=1$. The properties of the gamma-function imply

$$\begin{aligned}\frac{i^{s-1}\hat{z}^{1-s}}{\Gamma(s)} &= e^{(s-1)\log i}e^{-(s-1)\log \hat{z}}\Gamma^{-1}(s)\\ &= \bigl(1+(s-1)\log i+\cdots\bigr)\bigl(1-(s-1)\log \hat{z}+\cdots\bigr)\bigl(1+\gamma_0(s-1)+\cdots\bigr)\\ &= 1+(\gamma_0+\log i-\log \hat{z})(s-1)+\cdots. \end{aligned} \tag{5.5}$$

Therefore, from (5.3)–(5.5), we find that the main part of the Laurent series expansion for $\mathcal{Z}_{13}(s,\chi)$ at the point $s=1$ is

$$\begin{aligned}\mathcal{Z}_{13}(s,\chi) &= \frac{i^b}{\sqrt{q}E(\chi)G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\\ &\quad\times\left(\frac{1}{(s-1)^2}+\frac{2\gamma_0+\log\frac{(q,a-1)^2}{2\pi q}}{s-1}\right)+\cdots. \end{aligned} \tag{5.6}$$

To find other poles of $I_{63}(s,\chi)$, we consider the terms in (5.2), and (5.3) with $k=1,2,3\dots$ having in mind that some of the coefficients $b_k(s)$ can vanish and cancel the possible poles. In [2], it was obtained that

$$b_j(1-j) = \frac{(-1)^j}{2\pi i}\int\limits_{|z|=r}\frac{\mathrm{d}z}{z^j(e^{z/2}-e^{-z/2})} = \frac{(-1)^j}{2\pi i}\int\limits_{|z|=r}\frac{\mathrm{d}z}{z^j 2\sinh\frac{z}{2}}, \tag{5.7}$$

where $\sinh(\frac{z}{2}) = \frac{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}{2}$. Therefore, $(-1)^j b_j(1-j)$ is the $(j-1)$ th coefficient of the Laurent series expansion for the function $\bigl(2\sinh\frac{z}{2}\bigr)^{-1}$ at the point $z=0$. It is well known that the Laurent series expansion of this function we can write using the Bernoulli numbers B_{2k}, namely,

$$\frac{1}{2\sinh\frac{z}{2}} = \frac{1}{z}-\sum_{k=1}^{\infty}\frac{(2^{2k-1}-1)B_{2k}}{2^{2k-1}(2k)!}z^{2k-1}. \tag{5.8}$$

In formula (5.8), we have only odd powers of z. From one hand, this means that the coefficients standing in front of the term $z^{2j}, j\in\mathbb{N}$, in this formula are equals to zero, therefore, $b_{2j+1}(-2j)=0, j\in\mathbb{N}_0$, and no simple poles at these points. From the other hand, the coefficient standing in front of the term $z^{2j-1}, j\in\mathbb{N}$, in formula (5.8) corresponds the coefficient $2j, j\in\mathbb{N}$, in formula (5.7), and we have simple poles at the points $1-2j, j\in\mathbb{N}$, because even Bernoulli numbers do not vanish, therefore, $b_{2j}(1-2j)\neq 0$. Clearly,

$$(-1)^{2j}b_{2j}(1-2j) = -\frac{(1-2^{-(2j-1)})B_{2j}}{(2j)!}. \tag{5.9}$$

With purpose to get residues at points $1-2j, j\in\mathbb{N}$, we consider the function

$$\begin{aligned}J(s) &\stackrel{\text{def}}{=} \frac{2\pi i^b}{\Gamma(s)\sqrt{q}E(\chi)}\sum_{k=1}^{\infty}d(k)\chi(k)e^{-\frac{2\pi i k}{q}}\\ &\quad\times\int_0^{\infty}e^{-\frac{2\pi k z}{q}}\left(\sum_{j=0}^{l}b_j(s)(-iz)^j\right)z^{s-1}\mathrm{d}z, \end{aligned}$$

which get non-holomorphic part of the Mellin transform. In the same way as in [2], taking into account Lemma 3, we write $J(s)$ in the form of finite sum of Estermann zeta-functions

$$J(s) = \frac{2\pi i^b}{\sqrt{q}E(\chi)} \sum_{j=0}^{l} (-i)^j b_j(s) \left(\frac{q}{2\pi}\right)^{s+j} s(s+1)\ldots(s+j-1)$$
$$\times \frac{1}{G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) E\left(s+j; \frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}}, 0\right).$$

Then, using (3.1)) and (5.9) we find, that

$$\operatorname*{Res}_{\substack{s=1-2j \\ j\in\mathbb{N}}} \mathcal{Z}_1(s,\chi) = \frac{i^{b-2j}(1-2^{-(2j-1)})B_{2j}}{\sqrt{q}E(\chi)G(\overline{\chi})2j} \sum_{a=1}^{q} \overline{\chi}(a)(q,a-1). \tag{5.10}$$

Using properties of Dirichlet characters, we write the Gauss sum in the form

$$G(\chi) = \sum_{l=1}^{q} \chi(l) e^{2\pi i l/q} = \sum_{l=0}^{q-1} \chi(l) e^{2\pi i l/q}.$$

Then we have

$$\begin{aligned} \chi(-1)\overline{G(\chi)} &= \chi(-1) \sum_{l=1}^{q} \overline{\chi}(l) e^{-2\pi i l/q} \\ &= \chi(-1) \sum_{l=1}^{q} \overline{\chi}(-1)\overline{\chi}(-l) e^{-2\pi i l/q} = \left|\chi(-1)\right|^2 \sum_{l=1}^{q} \overline{\chi}(-l) e^{-2\pi i l/q} \\ &= \sum_{l=1}^{q} \overline{\chi}(q-l) e^{2\pi i (q-l)/q} = \sum_{m=0}^{q-1} \overline{\chi}(m) e^{2\pi i m/q} = G(\overline{\chi}). \end{aligned}$$

Also, it is well known that $\left|G(\chi)\right|^2 = q$. Thus,

$$\frac{i^b}{\sqrt{q}E(\chi)G(\overline{\chi})} = \frac{i^b}{\sqrt{q}(-1)^b \frac{G(\chi)}{\sqrt{q}} \chi(-1)\overline{G(\chi)}} = \frac{(-i)^b}{\chi(-1)|G(\chi)|^2} = \frac{(-i)^b}{(-1)^b q} = \frac{i^b}{q}.$$

Now this, (5.6), and (5.10) give the assertion of the theorem.

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