

A Limit Theorem for Twists of L -Functions of Elliptic Curves

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Abstract. In the paper, a limit theorem for weakly convergent probability measures on the complex plane for twisted with Dirichlet character L -functions of elliptic curves with an increasing modulus of the character is proved.

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1 Introduction

The present paper is a continuation of [5] and [7]. Therefore, we briefly remind the limit theorems for the modulus and argument for twists of L -functions of elliptic curves obtained in [5] and [4], respectively.

Let E be an elliptic curve over the field of rational numbers given by the Weierstrass equation

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

with non-zero discriminant $\Delta = -16(4a^3 + 27b^2)$. For a prime number p , denote by E_p the reduction of the curve E modulo p which is a curve over the finite field \mathbb{F}_p , and define $\lambda(p)$ by

$$|E(\mathbb{F}_p)| = p + 1 - \lambda(p),$$

where $|E(\mathbb{F}_p)|$ denotes the number of points of E_p . Let $s = \sigma + it$ be a complex variable. Then the L -function of the elliptic curve E is defined, for $\sigma > \frac{3}{2}$, by

$$L_E(s) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p \nmid \Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}.$$

In view of the Taniyama-Shimura conjecture which was proved in [1], the function $L_E(s)$ continues analytically to an entire function.

Now let χ be a Dirichlet character modulo q . Then the twist $L_E(s, \chi)$ of $L_E(s)$ with the character χ is defined, for $\sigma > \frac{3}{2}$, by

$$L_E(s, \chi) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right)^{-1} \prod_{p \nmid \Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s} + \frac{\chi^2(p)}{p^{2s-1}}\right)^{-1},$$

and continues analytically to an entire function. Moreover, the function $L_E(s)$, for $\sigma > \frac{3}{2}$, can be rewritten in the form

$$L_E(s, \chi) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right)^{-1} \prod_{p \nmid \Delta} \left(1 - \frac{\alpha(p)\chi(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\chi(p)}{p^s}\right)^{-1}, \tag{1.1}$$

where $\alpha(p)$ and $\beta(p)$ are complex conjugate numbers satisfying $\alpha(p)\beta(p) = p$ and $\alpha(p) + \beta(p) = \lambda(p)$.

Define two multiplicative functions $a_z(m)$ and $b_z(m)$, $m \in \mathbb{N}$, where, for $p \nmid \Delta$ and $l \in \mathbb{N}$,

$$a_z(p^l) = \sum_{j=0}^l d_z(p^j) \alpha^j(p) d_z(p^{l-j}) \beta^{l-j}(p),$$

$$b_z(p^l) = \sum_{j=0}^l d_z(p^j) \bar{\alpha}^j(p) d_z(p^{l-j}) \bar{\beta}^{l-j}(p),$$

while, for $p | \Delta$ and $l \in \mathbb{N}$,

$$a_z(p^l) = b_z(p^l) = d_z(p^l) \lambda^l(p),$$

where

$$d_z(p^l) = \frac{\theta(\theta + 1) \cdots (\theta + l - 1)}{l!} \quad \text{with} \quad \theta = \frac{z}{2}.$$

Suppose that the modulus q of the character χ is a prime number and is not fixed, and put, for $Q \geq 2$,

$$\mu_Q(\dots) = M_Q^{-1} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} 1, \quad \text{where} \quad M_Q = \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} 1,$$

χ_0 , as usual, denotes the principal character, and in place of dots a condition satisfied by a pair $(q, \chi(\text{mod } q))$ is to be written.

Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S , and, on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, define the probability measure $P_{\mathbb{R}}$ by the characteristic transforms

$$w_k(\tau) = \int_{\mathbb{R} \setminus \{0\}} |x|^{i\tau} \text{sgn}^k \, dP_{\mathbb{R}} = \sum_{m=1}^{\infty} \frac{a_{i\tau}(m) b_{i\tau}(m)}{m^{2\sigma}}, \quad \tau \in \mathbb{R}, \quad k = 0, 1.$$

Then a limit theorem for $|L_E(s, \chi)|$ proved in [5] is of the form.

Theorem 1. *Suppose that $\sigma > \frac{3}{2}$. Then $\mu_Q(|L_E(s, \chi)| \in A)$, $A \in \mathcal{B}(\mathbb{R})$, converges weakly to the measure $P_{\mathbb{R}}$ as $Q \rightarrow \infty$.*

In [3], a limit theorem for the argument of the function $L_E(s, \chi)$ has been proved. Denote by γ the unit circle on the complex plane, and let P_γ be a probability measure on $(\gamma, \mathcal{B}(\gamma))$ defined by the Fourier transform

$$g(k) = \int_{\gamma} x^k dP_\gamma = \sum_{m=1}^{\infty} \frac{a_k(m)b_{-k}(m)}{m^{2\sigma}}, \quad k \in \mathbb{Z}.$$

Then the main result of [4] is the following statement.

Theorem 2. *Suppose that $\sigma > \frac{3}{2}$. Then*

$$\mu_Q(\exp\{i \arg L_E(s, \chi)\} \in A), \quad A \in \mathcal{B}(\gamma),$$

converges weakly to the measure P_γ as $Q \rightarrow \infty$.

The aim of this note is to consider the weak convergence of the frequency

$$P_{Q, \mathbb{C}}(A) \stackrel{\text{def}}{=} \mu_Q(L_E(s, \chi) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

as $Q \rightarrow \infty$. In other words, we will obtain a limit theorem for the twists $L_E(s, \chi)$ as $Q \rightarrow \infty$ on the complex plane \mathbb{C} . This theorem joins Theorems 1 and 2.

Let, for brevity,

$$\theta_{\pm} = \theta(\tau, \pm k) = \frac{i\tau \pm k}{2}, \quad \hat{\theta}_{\pm} = 2\theta_{\pm}, \quad \tau \in \mathbb{R}, \quad k \in \mathbb{Z}.$$

On $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, define the probability measure $P_{\mathbb{C}}$ by the characteristic transform

$$w_k(\tau, k) = \int_{\mathbb{C} \setminus \{0\}} |z|^{i\tau} e^{ik \arg z} dP_{\mathbb{C}} = \sum_{m=1}^{\infty} \frac{a_{\hat{\theta}_+}(m)b_{\hat{\theta}_-}(m)}{m^{2\sigma}}, \quad \tau \in \mathbb{R}, \quad k \in \mathbb{Z}.$$

Theorem 3. *Suppose that $\sigma > \frac{3}{2}$. Then $P_{Q, \mathbb{C}}$ converges weakly to the measure $P_{\mathbb{C}}$ as $Q \rightarrow \infty$.*

2 Characteristic Transform of $P_{Q, \mathbb{C}}$

For the proof of Theorem 3, we apply the method of characteristic transforms of probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. Thus, we start with a formula for the characteristic transform $w_Q(\tau, k)$ of the measure $P_{Q, \mathbb{C}}$. By the definition of $P_{Q, \mathbb{C}}$, we have that

$$\begin{aligned} w_Q(\tau, k) &= \int_{\mathbb{C} \setminus \{0\}} |z|^{i\tau} e^{ik \arg z} dP_{Q, \mathbb{C}} \\ &= \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\pmod q) \\ \chi \neq \chi_0}} |L_E(s, \chi)|^{i\tau} e^{ik \arg L_E(s, \chi)}, \quad \tau \in \mathbb{R}, \quad k \in \mathbb{Z}. \end{aligned} \tag{2.1}$$

For arbitrary $\delta > 0$, denote by R the half-plane $\{s \in \mathbb{C} : \sigma \geq \frac{3}{2} + \delta\}$. Then, in view of the Hasse estimate

$$|\lambda(p)| \leq 2\sqrt{p}$$

and (1.1), we have that $L_E(s, \chi) \neq 0$ for $s \in R$. Thus,

$$|L_E(s, \chi)| = (L_E(s, \chi)\overline{L_E(s, \chi)})^{\frac{1}{2}}$$

and

$$e^{i\arg L_E(s, \chi)} = \left(\frac{L_E(s, \chi)}{\overline{L_E(s, \chi)}}\right)^{\frac{1}{2}}.$$

Using the Euler product (1.1), we find that, for $s \in R$,

$$\begin{aligned} & |L_E(s, \chi)|^{i\tau} e^{ik \arg L(s, \chi)} \\ &= \exp \left\{ -\frac{i\tau}{2} \sum_{p|\Delta} \left(\log \left(1 - \frac{\lambda(p)\chi(p)}{p^s} \right) + \log \left(1 - \frac{\lambda(p)\overline{\chi}(p)}{p^{\overline{s}}} \right) \right) \right. \\ &\quad - \frac{k}{2} \sum_{p|\Delta} \left(\log \left(1 - \frac{\lambda(p)\chi(p)}{p^s} \right) - \log \left(1 - \frac{\lambda(p)\overline{\chi}(p)}{p^{\overline{s}}} \right) \right) \\ &\quad - \frac{i\tau}{2} \sum_{p \nmid \Delta} \left(\log \left(1 - \frac{\alpha(p)\chi(p)}{p^s} \right) + \log \left(1 - \frac{\beta(p)\chi(p)}{p^s} \right) \right) \\ &\quad - \frac{i\tau}{2} \sum_{p \nmid \Delta} \left(\log \left(1 - \frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}} \right) + \log \left(1 - \frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}} \right) \right) \\ &\quad - \frac{k}{2} \sum_{p \nmid \Delta} \left(\log \left(1 - \frac{\alpha(p)\chi(p)}{p^s} \right) + \log \left(1 - \frac{\beta(p)\chi(p)}{p^s} \right) \right) \\ &\quad \left. + \frac{k}{2} \sum_{p \nmid \Delta} \left(\log \left(1 - \frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}} \right) + \log \left(1 - \frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}} \right) \right) \right\} \\ &= \prod_{p|\Delta} \exp \left\{ -\frac{i\tau}{2} \log \left(1 - \frac{\lambda(p)\chi(p)}{p^s} \right) - \frac{k}{2} \log \left(1 - \frac{\lambda(p)\chi(p)}{p^s} \right) \right\} \\ &\quad \times \prod_{p|\Delta} \exp \left\{ -\frac{i\tau}{2} \log \left(1 - \frac{\lambda(p)\overline{\chi}(p)}{p^{\overline{s}}} \right) + \frac{k}{2} \log \left(1 - \frac{\lambda(p)\overline{\chi}(p)}{p^{\overline{s}}} \right) \right\} \\ &\quad \times \prod_{p \nmid \Delta} \exp \left\{ -\frac{i\tau}{2} \left(\log \left(1 - \frac{\alpha(p)\chi(p)}{p^s} \right) + \log \left(1 - \frac{\beta(p)\chi(p)}{p^s} \right) \right) \right. \\ &\quad \left. - \frac{k}{2} \left(\log \left(1 - \frac{\alpha(p)\chi(p)}{p^s} \right) + \log \left(1 - \frac{\beta(p)\chi(p)}{p^s} \right) \right) \right\} \\ &\quad \times \prod_{p \nmid \Delta} \exp \left\{ -\frac{i\tau}{2} \left(\log \left(1 - \frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}} \right) + \log \left(1 - \frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}} \right) \right) \right. \\ &\quad \left. + \frac{k}{2} \left(\log \left(1 - \frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}} \right) + \log \left(1 - \frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}} \right) \right) \right\} \end{aligned}$$

$$\begin{aligned}
 &= \prod_{p|\Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right)^{-\theta_+} \prod_{p|\Delta} \left(1 - \frac{\lambda(p)\bar{\chi}(p)}{p^{\bar{s}}}\right)^{-\theta_-} \\
 &\times \prod_{p\nmid\Delta} \left(1 - \frac{\alpha(p)\chi(p)}{p^s}\right)^{-\theta_+} \left(1 - \frac{\beta(p)\chi(p)}{p^s}\right)^{-\theta_+} \\
 &\times \prod_{p\nmid\Delta} \left(1 - \frac{\bar{\alpha}(p)\bar{\chi}(p)}{p^{\bar{s}}}\right)^{-\theta_-} \left(1 - \frac{\bar{\beta}(p)\bar{\chi}(p)}{p^{\bar{s}}}\right)^{-\theta_-}. \tag{2.2}
 \end{aligned}$$

Here the multi-valued functions $\log(1 - z)$ and $(1 - z)^{-w}$, $w \in \mathbb{C} \setminus \{0\}$, in the region $|z| < 1$ are defined by continuous variation along any path lying in this region from the values $\log(1 - z)|_{z=0} = 0$ and $(1 - z)^{-w}|_{z=0} = 1$, respectively.

In the above notation, we have that, for $|z| < 1$,

$$(1 - z)^{-\theta_{\pm}} = \sum_{l=0}^{\infty} d_{\hat{\theta}_{\pm}}(p^l) z^l.$$

Therefore, equality (2.2) shows that, for $s \in R$,

$$\begin{aligned}
 &|L_E(s, \chi)|^{i\tau} e^{i\text{karg}L_E(s, \chi)} \\
 &= \prod_{p|\Delta} \sum_{j=0}^{\infty} \frac{d_{\hat{\theta}_+}(p^j) \lambda^j(p) \chi^j(p)}{p^{js}} \\
 &\times \prod_{p|\Delta} \sum_{j=0}^{\infty} \frac{d_{\hat{\theta}_-}(p^j) \lambda^j(p) \bar{\chi}^j(p)}{p^{j\bar{s}}} \prod_{p\nmid\Delta} \sum_{l=0}^{\infty} \frac{d_{\hat{\theta}_+}(p^l) \alpha^l(p) \chi^l(p)}{p^{ls}} \\
 &\times \prod_{p\nmid\Delta} \sum_{r=0}^{\infty} \frac{d_{\hat{\theta}_+}(p^r) \beta^r(p) \chi^r(p)}{p^{rs}} \prod_{p\nmid\Delta} \sum_{l=0}^{\infty} \frac{d_{\hat{\theta}_-}(p^l) \bar{\alpha}^l(p) \bar{\chi}^l(p)}{p^{l\bar{s}}} \\
 &\times \prod_{p\nmid\Delta} \sum_{r=0}^{\infty} \frac{d_{\hat{\theta}_-}(p^r) \bar{\beta}^r(p) \bar{\chi}^r(p)}{p^{r\bar{s}}}. \tag{2.3}
 \end{aligned}$$

Let $\hat{a}_z(m)$ and $\hat{b}_z(m)$ be multiplicative functions with respect to m defined, for $p \nmid \Delta$ and $l \in \mathbb{N}$, by

$$\hat{a}_z(p^l) = \sum_{j=0}^l d_z(p^j) \alpha^j(p) \chi^j(p) d_z(p^{l-j}) \beta^{l-j}(p) \chi^{l-j}(p) \tag{2.4}$$

and

$$\hat{b}_z(p^l) = \sum_{j=0}^l d_z(p^j) \bar{\alpha}^j(p) \bar{\chi}^j(p) d_z(p^{l-j}) \bar{\beta}^{l-j}(p) \bar{\chi}^{l-j}(p), \tag{2.5}$$

and, for $p | \Delta$ and $l \in \mathbb{N}$, by

$$\hat{a}_z(p^l) = d_z(p^l) \lambda^l(p) \chi^l(p) \tag{2.6}$$

and

$$\hat{b}_z(p^l) = d_z(p^l)\lambda^l(p)\bar{\chi}^l(p). \tag{2.7}$$

For $|\tau| \leq c$, with arbitrary $c > 0$, and $l \in \mathbb{N}$, we have that

$$\begin{aligned} |d_{\hat{\theta}_\pm}(p^l)| &\leq \frac{|\theta_\pm|(|\theta_\pm| + 1)\dots(|\theta_\pm| + l - 1)}{l!} \\ &\leq |\theta_\pm| \prod_{v=1}^l \left(1 + \frac{|\theta_\pm|}{v}\right) \leq (l + 1)^{c_1}, \end{aligned} \tag{2.8}$$

where c_1 depends on c and k , only. By the definition of the quantities $\alpha(p)$ and $\beta(p)$, we find that $|\alpha(p)| = |\beta(p)| = \sqrt{p}$. Thus, for $p \nmid \Delta$ and $l \in \mathbb{N}$, equalities (2.4) and (2.5) give the bounds

$$|\hat{a}_{\hat{\theta}_+}(p^l)| \leq p^{\frac{l}{2}} \sum_{j=0}^l (j + 1)^{c_1} (l - j + 1)^{c_1} \leq p^{\frac{l}{2}} (l + 1)^{2c_1+1} \tag{2.9}$$

and

$$|\hat{b}_{\hat{\theta}_-}(p^l)| \leq p^{\frac{l}{2}} (l + 1)^{2c_1+1}. \tag{2.10}$$

It is known [6] that, for $p \mid \Delta$, the numbers $\lambda(p)$ take the values 1 or 0. Therefore, in view of (2.6)–(2.8), for $p \mid \Delta$ and $l \in \mathbb{N}$,

$$|\hat{a}_{\hat{\theta}_+}(p^l)| \leq (l + 1)^{c_1}, \quad |\hat{b}_{\hat{\theta}_-}(p^l)| \leq (l + 1)^{c_1}. \tag{2.11}$$

Now estimates (2.9)–(2.11) and the multiplicativity of the functions $\hat{a}_z(m)$ and $\hat{b}_z(m)$ yield

$$|\hat{a}_{\hat{\theta}_\pm}(m)| = \sqrt{m} \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} (l + 1)^{2c_1+1} = \sqrt{m} d^{2c_1+1}(m), \tag{2.12}$$

$$|\hat{b}_{\hat{\theta}_-}(m)| \leq \sqrt{m} d^{2c_1+1}(m), \tag{2.13}$$

where $d(m)$ denotes the divisor function. Since

$$d(m) = O_\varepsilon(m^\varepsilon) \tag{2.14}$$

with every $\varepsilon > 0$, the above estimates show that the series

$$\sum_{m=1}^\infty \frac{a_{\hat{\theta}_+}(m)}{m^s} \quad \text{and} \quad \sum_{m=1}^\infty \frac{b_{\hat{\theta}_-}(m)}{m^{\bar{s}}}$$

converge absolutely for $|\tau| \leq c$, $k \in \mathbb{Z}$ and $s \in R$. Therefore, taking into account (2.3), we obtain that, for $|\tau| \leq c$, $k \in \mathbb{Z}$ and $s \in R$,

$$\begin{aligned} &|L_E(s, \chi)|^{i\tau} e^{ik \arg L_E(s, \chi)} \\ &= \prod_{p \mid \Delta} \sum_{j=0}^\infty \frac{\hat{a}_{\hat{\theta}_+}(p^j)}{p^{js}} \prod_{p \mid \Delta} \sum_{l=0}^\infty \frac{\hat{b}_{\hat{\theta}_-}(p^l)}{p^{l\bar{s}}} \prod_{p \nmid \Delta} \sum_{j=0}^\infty \frac{\hat{a}_{\hat{\theta}_+}(p^j)}{p^{js}} \prod_{p \nmid \Delta} \sum_{l=0}^\infty \frac{\hat{b}_{\hat{\theta}_-}(p^l)}{p^{l\bar{s}}} \\ &= \sum_{m=1}^\infty \frac{\hat{a}_{\hat{\theta}_+}(m)}{m^s} \sum_{n=1}^\infty \frac{\hat{b}_{\hat{\theta}_-}(n)}{n^{\bar{s}}}. \end{aligned}$$

Substituting this into (2.1), we find that

$$w_Q(\tau, k) = \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\pmod q) \\ \chi \neq \chi_0}} \sum_{m=1}^{\infty} \frac{\hat{a}_{\hat{\theta}_+}(m)}{m^s} \sum_{n=1}^{\infty} \frac{\hat{b}_{\hat{\theta}_-}(n)}{n^{\bar{s}}}. \tag{2.15}$$

3 Proof of Theorem 3

For the proof of Theorem 3, we apply the following continuity theorem in terms of characteristic transforms for probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. Let $P_n, n \in \mathbb{N}$, and P be probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. We say that P_n , as $n \rightarrow \infty$, converges weakly in the sense of \mathbb{C} to P if P_n converges weakly to P as $n \rightarrow \infty$, and, additionally,

$$\lim_{n \rightarrow \infty} P_n(\{0\}) = P(\{0\}).$$

Denote by $w_n(\tau, k)$ the characteristic transform of the measure P_n .

Lemma 1. *Suppose that*

$$\lim_{n \rightarrow \infty} w_n(\tau, k) = w(\tau, k), \quad \tau \in \mathbb{R}, \quad k \in \mathbb{Z},$$

where the function $w(\tau, 0)$ is continuous at the point $\tau = 0$. Then, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure P such that P_n converges weakly in the sense of \mathbb{C} to P as $n \rightarrow \infty$. In this case, $w(\tau, k)$ is the characteristic transform of the measure P .

Proof of the lemma is given in [7], [8].

Proof of Theorem 3. Using formula (2.15), we will obtain the asymptotics for $w_Q(\tau, k)$ as $Q \rightarrow \infty$. Let $r = \log Q$. Then estimates (2.12)–(2.14) show that, uniformly in $s \in R, |\tau| \leq c$, and, for any fixed $k \in \mathbb{Z}$ and $\varepsilon > 0$,

$$\sum_{m > r} \frac{\hat{a}_{\hat{\theta}_+}(m)}{m^s} = O_\varepsilon \left(\sum_{m > r} \frac{1}{m^{1+\delta-\varepsilon}} \right) = O_\varepsilon(r^{-\delta+\varepsilon})$$

and

$$\sum_{m > r} \frac{\hat{b}_{\hat{\theta}_-}(m)}{m^{\bar{s}}} = O_\varepsilon(r^{-\delta+\varepsilon}).$$

Substituting this in (2.15), we obtain that, for $s \in R, |\tau| \leq c$ and any fixed $k \in \mathbb{Z}$,

$$w_Q(\tau, k) = \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\pmod q) \\ \chi \neq \chi_0}} \left(\left(\sum_{m \leq r} \frac{\hat{a}_{\hat{\theta}_+}(m)}{m^s} + O_\varepsilon(r^{-\delta+\varepsilon}) \right) \left(\sum_{n \leq r} \frac{\hat{b}_{\hat{\theta}_-}(n)}{n^{\bar{s}}} + O_\varepsilon(r^{-\delta+\varepsilon}) \right) \right)$$

$$\begin{aligned}
 &= \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\pmod q) \\ \chi \neq \chi_0}} \left(\sum_{m \leq r} \frac{\hat{a}_{\hat{\theta}_+}(m)}{m^s} \sum_{n \leq r} \frac{\hat{b}_{\hat{\theta}_-}(n)}{n^{\bar{s}}} \right) \\
 &+ O_\varepsilon \left(r^{-\delta+\varepsilon} \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\pmod q) \\ \chi \neq \chi_0}} \left(\left| \sum_{m \leq r} \frac{\hat{a}_{\hat{\theta}_+}(m)}{m^s} \right| \left| \sum_{n \leq r} \frac{\hat{b}_{\hat{\theta}_-}(n)}{n^{\bar{s}}} \right| \right) \right) \\
 &+ O_\varepsilon(r^{-\delta+\varepsilon}) \\
 &= \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\pmod q) \\ \chi \neq \chi_0}} \left(\sum_{m \leq r} \frac{\hat{a}_{\hat{\theta}_+}(m)}{m^s} \sum_{n \leq r} \frac{\hat{b}_{\hat{\theta}_-}(n)}{n^{\bar{s}}} \right) + O_\varepsilon(r^{-\delta+\varepsilon}). \tag{3.1}
 \end{aligned}$$

Here we have used the estimates

$$\sum_{m \leq r} \frac{\hat{a}_{\hat{\theta}_+}(m)}{m^s} = O \left(\sum_{m \leq r} \frac{1}{m^{1+\delta-\varepsilon}} \right) = O(1), \quad \sum_{n \leq r} \frac{\hat{b}_{\hat{\theta}_-}(n)}{n^{\bar{s}}} = O(1),$$

which are uniform in $s \in R$ and $|\tau| \leq c$.

By (2.4)–(2.7), using the multiplicativity of $\hat{a}_{\hat{\theta}_+}(m)$ and $\hat{b}_{\hat{\theta}_-}(n)$, the complete multiplicativity of Dirichlet characters as well as the notation for $a_{\hat{\theta}_+}(m)$ and $b_{\hat{\theta}_-}(n)$, we find that

$$\begin{aligned}
 \hat{a}_{\hat{\theta}_+}(m) &= \prod_{\substack{p^l | m \\ p^{l+1} \nmid m, p \nmid \Delta}} \sum_{j=0}^l d_{\hat{\theta}_+}(p^j) \alpha^j(p) d_{\hat{\theta}_+}(p^{l-j}) \beta^{l-j}(p) \chi(p^l) \\
 &\times \prod_{\substack{p^l | m \\ p^{l+1} \nmid m, p | \Delta}} d_{\hat{\theta}_+}(p^l) \lambda^l(p) \chi(p^l) \\
 &= \chi(m) \prod_{\substack{p^l | m \\ p^{l+1} \nmid m, p \nmid \Delta}} \sum_{j=0}^l d_{\hat{\theta}_+}(p^j) \alpha^j(p) d_{\hat{\theta}_+}(p^{l-j}) \beta^{l-j}(p) \\
 &\times \prod_{\substack{p^l | m \\ p^{l+1} \nmid m, p | \Delta}} d_{\hat{\theta}_+}(p^l) \lambda^l(p) = a_{\hat{\theta}_+}(m) \chi(m), \quad \hat{b}_{\hat{\theta}_-}(m) = b_{\hat{\theta}_-}(m) \bar{\chi}(m).
 \end{aligned}$$

Thus, (3.1) can be written in the form

$$\begin{aligned}
 w_Q(\tau, k) &= \sum_{m \leq r} \frac{a_{\hat{\theta}_+}(m)}{m^s} \sum_{n \leq r} \frac{b_{\hat{\theta}_-}(n)}{n^{\bar{s}}} \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\pmod q) \\ \chi \neq \chi_0}} \chi(m) \bar{\chi}(n) \\
 &+ O_\varepsilon(r^{-\delta+\varepsilon}). \tag{3.2}
 \end{aligned}$$

First we suppose that $m \neq n$. Then, clearly,

$$\begin{aligned} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\pmod q) \\ \chi \neq \chi_0}} \chi(m)\bar{\chi}(n) &= \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\pmod q) \\ \chi \neq \chi_0}} |\chi(m)|^2 \\ &= M_Q - \sum_{\substack{q|m \\ q \leq r}} (q-2) = M_Q + O\left(\sum_{q \leq r} q\right) = M_Q + O(r^2). \end{aligned} \tag{3.3}$$

If $m \neq n$, then, in view of the formula

$$\sum_{\chi = \chi(\pmod q)} \chi(m)\bar{\chi}(n) = \begin{cases} q-1 & \text{if } m \equiv n \pmod q, \\ 0 & \text{if } m \not\equiv n \pmod q \end{cases}$$

with $(m, q) = 1$, we find that

$$\begin{aligned} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\pmod q) \\ \chi \neq \chi_0}} \chi(m)\bar{\chi}(n) &= \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\pmod q) \\ q|(m-n)}} \chi(m)\bar{\chi}(n) \\ &\quad + \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\pmod q) \\ q \nmid (m-n)}} \chi(m)\bar{\chi}(n) + O\left(\sum_{q \leq Q} 1\right) \\ &= O\left(\sum_{q \leq r} q\right) + O\left(\frac{Q}{\log Q}\right) = O\left(\frac{Q}{\log Q}\right). \end{aligned}$$

Therefore, in view of the estimate [8]

$$M_Q = \frac{Q^2}{2 \log Q} + O\left(\frac{Q^2}{\log^2 Q}\right),$$

we obtain from (3.2) and (3.3) that, uniformly in $s \in R$, $|\tau| \leq c$, and any fixed $k \in \mathbb{Z}$

$$w_Q(\tau, k) = \sum_{m=1}^{\infty} \frac{a_{\hat{\theta}_+}(m)b_{\hat{\theta}_-}(m)}{m^{2\sigma}} + o(1) \tag{3.4}$$

as $Q \rightarrow \infty$.

The functions $a_{\hat{\theta}_+}(m)$ and $b_{\hat{\theta}_-}(m)$ are continuous in τ . Therefore, the uniform convergence for $|\tau| \leq c$ of the series

$$w(\tau, k) = \sum_{m=1}^{\infty} \frac{a_{\hat{\theta}_+}(m)b_{\hat{\theta}_-}(m)}{m^{2\sigma}}$$

shows that the function $w(\tau, 0)$ is continuous at $\tau = 0$. Therefore, (3.4) together with Lemma 1 proves the theorem. \square

4 Concluding Remarks

Denote by $L(s, \chi)$ the Dirichlet L -functions. E. Stankus in [9] obtained, for $\sigma > \frac{1}{2}$, the weak convergence for

$$P_Q(A) \stackrel{def}{=} \mu_Q(L_E(s, \chi) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

as $Q \rightarrow \infty$. Let P be a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ defined by the characteristic transform

$$w_R(\tau, k) = \sum_{m=1}^{\infty} \frac{d_{\hat{\theta}_+}(m)d_{\hat{\theta}_-}(m)}{m^{2\sigma}}, \quad \sigma > \frac{1}{2}, \quad \tau \in \mathbb{R}, \quad k \in \mathbb{Z}_0.$$

Then he proved that P_Q converges weakly to P as $Q \rightarrow \infty$. This theorem connects limit theorems for $|L(s, \chi)|$ and $\arg L(s, \chi)$ obtained by P.D.T.A. Elliott in [2] and [3], respectively.

Thus, a limit theorem for Dirichlet L -functions is valid also on the left of the absolute convergence half-plane $\sigma > 1$. For the proof of such a theorem, the convergence, for $\sigma > \frac{1}{2}$, of the series

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \chi \neq \chi_0$$

is essentially used. Differently from Dirichlet L -functions, we do not have any information on the convergence of the series for $L_E(s, \chi)$, $\chi \neq \chi_0$, in the half-plane $\sigma > 1$. Therefore, we can prove Theorem 3 only in the half-plane of absolute convergence of the series for $L_E(s, \chi)$. However, we conjecture that Theorem 3 remains also true for $\sigma > 1$.

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