A Truncation Method Based on Hermite Expansion for Unknown Source in Space Fractional Diffusion Equation

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Abstract. In this paper we consider the problem for identifying an unknown steady source in a space fractional diffusion equation. A truncation method based on a Hermite function expansion is proposed, and the regularization parameter is chosen by a discrepancy principle. An error estimate between the exact solution and its approximation is given. A numerical implementation is discussed and corresponding results are presented to verify the effectiveness of the method.

Keywords: ill-posed problem, unknown source, truncation method, Hermite functions expansion, discrepancy principle.

AMS Subject Classification: 65F22.

1 Introduction

In recent years, fractional calculus and fractional differential equations have attracted great attention in the scientific and engineering community. One of the most successful applications of fractional calculus and fractional differential equations is to effectively characterize the anomalous diffusion [11, 12]. The anomalous diffusion is described by the nonlinear growth of the mean square displacement $x(t)$ of a diffusion particle over time t: $\langle x^2(t) \rangle \sim \kappa_\alpha t^\alpha$, where κ_α is the diffusion coefficient, and α is the anomalous diffusion exponent. For different α, the anomalous diffusion is classified into subdiffusion ($0 < \alpha < 1$), normal diffusion ($\alpha = 1$), superdiffusion ($\alpha > 1$), and ballistic diffusion ($\alpha = 2$) [11, 18].

The space fractional diffusion equation we consider in this paper can describe the probability distribution of the particles having superdiffusion [18]

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial J^{\alpha}_{p,q}(x,t)}{\partial x} + f(x), \tag{1.1}$$

where

$$J^{\alpha}_{p,q}(x,t) = -\kappa_{\alpha}\{p_{-\infty}D^{\alpha-1}_{x} - q_{x}D^{\alpha-1}_{\infty}\}u$$

with $\kappa_{\alpha} > 0$, and $p + q = 1$ for $p,q \geq 0$. The fractional derivatives $_{-\infty}D^{\alpha}_{x}$ and $_{x}D^{\alpha}_{\infty}$ are left and right Riemann–Liouville fractional derivatives of order α $(1 < \alpha < 2)$ defined by

$$_{-\infty}D^{\alpha}_{x}u(x,t) = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_{-\infty}^{x}(x-\varsigma)^{1-\alpha}u(\varsigma,t)\,d\varsigma$$

and

$$_{x}D^{\alpha}_{\infty}u(x,t) = \frac{(-1)^{\lceil\alpha\rceil}}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_{x}^{\infty}(\varsigma-x)^{1-\alpha}u(\varsigma,t)\,d\varsigma,$$

such that $\lceil\alpha\rceil$ is the smallest integer no less than α (for $1 < \alpha < 2$, we have $\lceil\alpha\rceil = 2$).

Our goal is to identify the source term $f(x)$ in the equation (1.1) with the boundary conditions

$$u(x,0) = 0, \quad u(x,T) = g(x); \tag{1.2}$$

that is, to determine the source $f(x)$ in (1.1) from some observed boundary data in (1.2): $g \to f$. This problem is called the inverse source problem [9], and it is one of the most important problems in many branches of engineering sciences. Inverse source problems are well known to be ill posed and the data $g(x)$ usually contains error. Thus, the numerical simulation of these problems is very difficult and some special regularization is required.

Some results on the inverse problem of time fractional diffusion equations have been reported [3, 4, 10, 13]. However, there is limited work on space fractional diffusion equation [14, 18, 21]. There are some discussions on special cases of the inverse problem (1.1), namely $\alpha \to 2$, which refers to the sideways heat equation [2, 5, 6, 19, 24]. In the present paper, as an alternative way of dealing with the problem, we introduce a truncation method based on a Hermite expansion. The method is a very simple and effective method for solving some ill-posed problems and it has been successfully applied to numerical differentiation [23] and numerical analytic continuation [22]. We choose a regularization strategy based on a discrepancy principle explained in [20]. The theoretical analysis shows that the smoother the genuine solution, the higher the convergence rate of the numerical solution by our method.

This paper is organized as follows. Some preliminary materials are introduced in Section 2. In Section 3, the present method and corresponding convergence results are given. Some numerical tests are given in Section 4 to show the effectiveness of the new method.

2　Formulation of Problem and Solution

Let $\hat{h}$ denote the Fourier transform of $h \in L^2(\mathbb{R})$ defined by

$$\hat{h}(\xi) = \mathcal{F}[h(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x)e^{-i\xi x}\,dx$$

and $\|\cdot\|_s$ denotes the norm in Sobolev space $H^s(\mathbb{R})(s \geq 0)$ defined by

$$\|h\|_s := \left(\int_{\infty}^{\infty} \left(1 + \xi^2\right)^s |\hat{h}(\xi)|^2\,d\xi \right)^{1/2}.$$

When $s = 0$, $\|\cdot\|_0 =: \|\cdot\|$ denotes the $L^2(\mathbb{R})$ norm. It is well known that the Fourier transform of $_{-\infty}D_x^\alpha$ and $_xD_\infty^\alpha$ can be given as [15]

$$\mathcal{F}\{_{-\infty}D_x^\alpha u(x,t)\} = (i\xi)^\alpha \hat{u}(\xi,t), \qquad \mathcal{F}\{_xD_\infty^\alpha u(x,t)\} = (-i\xi)^\alpha \hat{u}(\xi,t), \quad \xi \in \mathbb{R}.$$

Then we have

$$\mathcal{F}\{J_{p,q}^\alpha(x,t)\} = \left[p(i\xi)^\alpha + q(-i\xi)^\alpha \right]\hat{u}(\xi,t), \quad \xi \in \mathbb{R}. \tag{2.1}$$

According to the definition of the complex power function, the principal value of the Fourier coefficient in (2.1) is [18]

$$p(i\xi)^\alpha + q(-i\xi)^\alpha = \left[\cos\left(\frac{\pi\alpha}{2}\right) + i(p-q)\operatorname{sgn}(\xi)\sin\left(\frac{\pi\alpha}{2}\right) \right] |\xi|^\alpha,$$

where $\operatorname{sgn}(x)$ is the signum function.

　　Application of the Fourier transform to problem (1.1) with respect to the variable x yields the following problem in the frequency space:

$$\begin{cases} \dfrac{\partial \hat{u}(\xi,t)}{\partial t} = \gamma_\alpha(\xi)\hat{u}(\xi,t) + \hat{f}(\xi), & \xi \in \mathbb{R},\ 0 < t < T, \\ \hat{u}(\xi,0) = 0, \quad \hat{u}(\xi,T) = \hat{g}(\xi), & \xi \in \mathbb{R}, \end{cases} \tag{2.2}$$

where

$$\gamma_\alpha(\xi) = \kappa_\alpha \cos(\frac{\pi\alpha}{2})|\xi|^\alpha + i[\kappa_\alpha(p-q)\sin(\pm\frac{\pi\alpha}{2})|\xi|^\alpha].$$

It is easy to see that the solution of problem (2.2) is

$$\hat{u}(\xi,t) = \frac{e^{\gamma_\alpha(\xi)t} - 1}{\gamma_\alpha(\xi)}\hat{f}(\xi).$$

So we get [18] $\hat{f}(\xi) = \lambda_\alpha(\xi)\hat{g}(\xi)$, where

$$\lambda_\alpha(\xi) = \frac{\gamma_\alpha(\xi)}{e^{\gamma_\alpha(\xi)T} - 1},$$

or equivalently

$$f = \mathcal{F}^{-1}\left[\lambda_\alpha(\xi)\hat{g}(\xi) \right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\gamma_\alpha(\xi)}{e^{\gamma_\alpha(\xi)T} - 1}\hat{g}(\xi)e^{i\xi x}\,d\xi =: \mathcal{T}g.$$

Obviously, the data $\hat{g}$ must decay faster than the rate $|\xi|^{-\alpha}$. However, in the practical applications, the data $g(x)$ is often obtained on the basis of reading of a physical instrument which is denoted by $g^\delta(x)$. In this case, such a decay is not likely to occur in the Fourier transform of the measured noisy data g^δ. So $\mathcal{T}g^\delta$ can not give a reliable approximation for f. In the following, a truncation method based on expanded Hermite functions is introduced to reconstruct a new function φ^δ from the perturbed data g^δ. $T\varphi^\delta$ will give a reliable approximation of f.

We first recall the definition of the Hermite functions and their well-known properties which are subsequently used. The normalized Hermite functions $h_\ell(x)$ are defined by

$$h_\ell(x) = \left(2^\ell \ell! \sqrt{\pi}\right)^{-1/2} e^{-\frac{1}{2}x^2} H_\ell(x),$$

where the $H_\ell(x)$ are the usual Hermite polynomials [17]. The set of Hermite functions is the $L^2(\mathbb{R})$-orthogonal system [16], i.e.,

$$\int_{\mathbb{R}} h_\ell(x) h_m(x)\, dx = \delta_{\ell,m},$$

where $\delta_{\ell,m}$ is the Dirac delta function. For any $\phi \in L^2(\mathbb{R})$, we may write $\phi(x) = \sum_{\ell=0}^{\infty} \phi_\ell h_\ell(x)$, where

$$\phi_\ell = \int_{\mathbb{R}} \phi(x) h_\ell(x)\, dx, \quad \ell = 0, 1, 2 \dots. \tag{2.3}$$

Let N be any positive integer and

$$\mathcal{H}_N = \text{span}\{h_0(x), h_1(x), \dots, h_N(x)\}.$$

We shall use c and c_α to denote two generic positive constants independent of any functions and the parameter N, but whose values may vary in each occurrence.

Lemma 1. [7] *For any $\varphi \in \mathcal{H}_N$ and $r \geq 0$,*

$$\|\varphi\|_r \leq cN^{\frac{r}{2}} \|\varphi\|.$$

The $L^2(\mathbb{R})$-orthogonal projection $\mathcal{P}_N : L^2(\mathbb{R}) \to \mathcal{H}_N$ is a mapping such that, for any $\phi \in L^2(\mathbb{R})$,

$$\langle \phi - \mathcal{P}_N \phi, \varphi \rangle = 0, \quad \forall \varphi \in \mathcal{H}_N,$$

or equivalently,

$$(\mathcal{P}_N \phi)(x) = \sum_{l=0}^{N} \phi_\ell h_\ell(x).$$

We define also

$$\mathcal{A}g(x) = \mathcal{D}_x g(x) + x g(x).$$

Analogously to [7], we introduce the space $\mathbb{H}_{\mathcal{A}}^m(\mathbb{R})$ defined by

$$\mathbb{H}_{\mathcal{A}}^m(\mathbb{R}) = \big\{v \mid v \text{ is measurable on } \mathbb{R} \text{ and } \|v\|_{m,\mathcal{A}} < \infty\big\}$$

and equipped with the norm $\|v\|_{m,\mathcal{A}} = \|\mathcal{A}^m v\|$. For any $r > 0$, the space $\mathbb{H}_{\mathcal{A}}^r(\mathbb{R})$ and its norm are defined by space interpolation [1].

Lemma 2. [7] *For any $g \in \mathbb{H}_{\mathcal{A}}^r(\mathbb{R})$ and $0 \le \mu \le r$,*

$$\|g - \mathcal{P}_N g\|_\mu \le c N^{\frac{\mu-r}{2}} \|g\|_{r,\mathcal{A}}.$$

Lemma 3. [17] *The Hermite functions are eigenfunctions of the Fourier transform*

$$\hat{h}_\ell(\xi) = (-i)^\ell h_\ell(\xi), \quad \ell = 1, 2, \dots.$$

Suppose that the exact data g and the measured data g^δ satisfy

$$\big\|g^\delta - g\big\| \le \delta, \qquad \big\|g^\delta\big\| > \tau\delta, \tag{2.4}$$

where $\delta > 0$ is a given constant called the error level and $\tau > 1$. In this case, the function $\varphi^{n,\delta}$ can be given as

$$g^\delta \to \varphi^{n,\delta} = \mathcal{P}_n g^\delta = \sum_{l=0}^{n} g_\ell^\delta h_\ell(x), \tag{2.5}$$

where g_ℓ^δ are produced using (2.3). The truncation parameter n plays an essential role in the accuracy of these approximations. In this paper, we make use of a discrepancy principle [20] as follows to choose an $n(\delta, g^\delta)$ a posteriori such that:

$$\big\|(I - \mathcal{P}_n)g^\delta\big\| \le \tau\delta < \big\|(I - \mathcal{P}_{n-1})g^\delta\big\|. \tag{2.6}$$

Then from Lemma 3

$$f^{n,\delta}(x) := \mathcal{T}\varphi^{n,\delta} = \mathcal{F}^{-1}\big[\lambda(\xi)\hat{\varphi}^{n,\delta}(\xi)\big] = \mathcal{F}^{-1}\bigg[\lambda(\xi) \sum_{\ell=0}^{n} g_\ell^\delta (-i)^\ell \hat{h}_\ell(\xi)\bigg] \tag{2.7}$$

is used as an approximation of $f(x)$.

3   Error Estimate of the Method

In this section, the error estimate of the truncation regularization method is derived. First we give some auxiliary results.

Lemma 4. [18] *For $1 < \alpha < 2$, there holds*

$$\big|\lambda_\alpha(\xi)\big| \le c_\alpha \big(1 + |\xi|^2\big)^{\frac{\alpha}{2}}.$$

Lemma 5. *For any $\varphi \in \mathcal{H}_N$ and $r \ge 0$,*

$$\|\mathcal{T}\varphi\|_r \le c_\alpha N^{\frac{r+\alpha}{2}} \|\varphi\|.$$

Proof. From Lemmas 1 and 4 and noting the fact that $\widehat{\mathcal{T}\varphi} = \lambda_\alpha \hat{\varphi}$,

$$
\begin{aligned}
\left\|\mathcal{T}\varphi\right\|_r &= \left(\int_{\mathbb{R}} \left(1 + \xi^2\right)^r \left|\lambda_\alpha(\xi)\hat{\varphi}(\xi)\right|^2 d\xi \right)^{1/2} \\
&\leq c_\alpha \left(\int_{\mathbb{R}} \left(1 + \xi^2\right)^{r+\alpha} \left|\hat{\varphi}(\xi)\right|^2 d\xi \right)^{1/2} = c_\alpha \|\varphi\|_{r+\alpha} \leq cN^{\frac{r+\alpha}{2}} \|\varphi\|. \qquad \square
\end{aligned}
$$

Lemma 6. *Given an $r \geq 0$, for any $g \in \mathbb{H}_{\mathcal{A}}^{r+\alpha}(\mathbb{R})$ and $0 \leq \mu \leq r$,*

$$
\left\|\mathcal{T}(I - \mathcal{P}_N)g\right\|_\mu \leq c_\alpha N^{\frac{\mu-r}{2}} \|g\|_{r+\alpha,\mathcal{A}}.
$$

Proof. From Lemmas 2 and 4

$$
\begin{aligned}
\left\|\mathcal{T}(I - \mathcal{P}_N)g\right\|_\mu &= \int_{\mathbb{R}} \left(1 + |\xi|^2\right)^\mu \left|\lambda_\alpha(\xi)\big(\mathcal{F}(I - \mathcal{P}_N)g\big)(\xi)\right|^2 d\xi \\
&\leq c_\alpha \left\|(I - \mathcal{P}_N)g\right\|_{\mu+\alpha} \leq c_\alpha N^{\frac{\mu+\alpha-r+\alpha}{2}} \|g\|_{r+\alpha,\mathcal{A}} \leq c_\alpha N^{\frac{\mu-r}{2}} \|g\|_{r+\alpha,\mathcal{A}}. \qquad \square
\end{aligned}
$$

Lemma 7. *If $\|\mathcal{T}\varphi\|_r$ is bounded, then we have*

$$
\|\mathcal{T}\varphi\| \leq \big(c_\alpha \|\varphi\|\big)^{\frac{r}{r+\alpha}} \|\mathcal{T}\varphi\|_r^{\frac{\alpha}{r+\alpha}}.
$$

Proof. By the Hölder inequality, Lemma 4 and noting that $\widehat{\mathcal{T}\varphi} = \lambda_\alpha \hat{\varphi}$

$$
\begin{aligned}
\|\mathcal{T}\varphi\|^2 &= \int_{\mathbb{R}} \left|\lambda_\alpha(\xi)\right|^2 |\hat{\varphi}|^2 d\xi = \int_{\mathbb{R}} \left|\lambda_\alpha(\xi)\right|^2 |\hat{\varphi}|^{\frac{2\alpha}{r+\alpha}} |\hat{\varphi}|^{\frac{2r}{r+\alpha}} d\xi \\
&\leq \left[\int_{\mathbb{R}} \left(|\lambda_\alpha(\xi)|^2 |\hat{\varphi}|^{\frac{2\alpha}{r+\alpha}}\right)^{\frac{r+\alpha}{\alpha}} d\xi \right]^{\frac{\alpha}{r+\alpha}} \left[\int_{\mathbb{R}} \left(|\hat{\varphi}|^{\frac{2r}{r+\alpha}}\right)^{\frac{r+\alpha}{r}} d\xi \right]^{\frac{r}{r+\alpha}} \\
&\leq \left[c_\alpha^{\frac{2r}{\alpha}} \int_{\mathbb{R}} \left(1 + \xi^2\right)^r |\widehat{\mathcal{T}\varphi}|^2 d\xi \right]^{\frac{\alpha}{r+\alpha}} \|\varphi\|^{\frac{2r}{r+\alpha}} = c_\alpha^{\frac{2r}{r+\alpha}} \|\varphi\|^{\frac{2r}{r+\alpha}} \|\mathcal{T}\varphi\|_r^{\frac{2\alpha}{r+\alpha}}. \qquad \square
\end{aligned}
$$

Theorem 1. *Suppose that the condition (2.4) holds for some g and g^δ, $g \in H_{\mathcal{A}}^{r+\alpha}(\mathbb{R})$. $f^{n,\delta}(x)$ is defined by (2.6) and (2.7), then*

$$
\left\|f^{n,\delta} - f\right\| = O\big(\delta^{\frac{r}{r+\alpha}}\big).
$$

Proof. By Lemmas 5 and 6,

$$
\begin{aligned}
\left\|f^{n,\delta} - f\right\|_r &= \left\|\mathcal{T}\varphi^{n,\delta} - \mathcal{T}g\right\|_r = \left\|\mathcal{T}\mathcal{P}_n g^\delta - \mathcal{T}\mathcal{P}_n g + \mathcal{T}\mathcal{P}_n g - \mathcal{T}g\right\|_r \\
&\leq \left\|\mathcal{T}\mathcal{P}_n(g^\delta - g)\right\|_r + \left\|\mathcal{T}(I - \mathcal{P}_n)g\right\|_r \leq cn^{\frac{r+\alpha}{2}}\delta + c\|g\|_{r+\alpha,\mathcal{A}}. \qquad (3.1)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\left\|\mathcal{P}_{n-1}g - g\right\| &= \left\|(\mathcal{P}_{n-1}g^\delta - g^\delta) - (I - \mathcal{P}_{n-1})(g - g^\delta)\right\| \\
&\geq \left| \left\|\mathcal{P}_{n-1}g^\delta - g^\delta\right\| - \left\|(I - \mathcal{P}_{n-1})(g - g^\delta)\right\| \right|.
\end{aligned}
$$

From (2.6), we have $\left\|\mathcal{P}_{n-1}g^\delta - g^\delta\right\| > \tau\delta$. And from (2.4)

$$
\left\|(I - \mathcal{P}_{n-1})(g - g^\delta)\right\| \leq \left\|(g - g^\delta)\right\| \leq \delta.
$$

So $\|\mathcal{P}_{n-1}g - g\| \geq (\tau - 1)\delta$. From Lemma 2

$$(\tau - 1)\delta \leq \|\mathcal{P}_{n-1}g - g\| \leq c(n-1)^{-\frac{r+\alpha}{2}}\|g\|_{r+\alpha,\mathcal{A}}.$$

Thus we can get

$$n \leq \left(\frac{c\|g\|_{r+\alpha,A}}{\tau - 1}\right)^{\frac{2}{r+\alpha}}\delta^{-\frac{2}{r+\alpha}} + 1. \tag{3.2}$$

By (3.1) and (3.2), there exists an bounded value $M_{\alpha,\delta}$ such that

$$\left\|f^{n,\delta} - f\right\|_r = \left\|\mathcal{T}\left(\varphi^{n,\delta} - g\right)\right\|_r \leq M_{\alpha,\delta}. \tag{3.3}$$

Moreover,

$$\left\|\varphi^{n,\delta} - g\right\| \leq \left\|\varphi^{n,\delta} - g^{\delta}\right\| + \left\|g^{\delta} - g\right\| \leq (\tau + 1)\delta. \tag{3.4}$$

The assertion of theorem can be obtained by (3.3), (3.4) and Lemma 7. □

4　Numerical Tests

We first discuss the algorithm implementation, and then some numerical tests is presented to verify the effectiveness of the method.

To reduce the impact of discretization errors, the discretization knots are chosen as $x_j = \sigma_{N,j}$, $j = 0, 1, \ldots, N$, where $\sigma_{N,j}$ $(0 \leq j \leq N)$ are the zeros of $H_{N+1}(x)$. We first get the data $g = \{g(x_j)\}_{j=0}^N$ representing values of $g(x)$ on the knots, then obtain the perturbation data g^{δ} as follows

$$g^{\delta}(x_i) = g(x_i) + \epsilon_i, \quad |\epsilon_i| < \delta_1,$$

where $\{\epsilon\}_{i=0}^N$ are generated by $(2 \times \text{rand}(N+1, 1) - 1) \times \delta_1$ in `Matlab`.

Let $\omega_{N,j}$ be the corresponding Hermite–Gauss weights, namely

$$\omega_{N,j} = \rho_{N,j}e^{\sigma_{N,j}^2},$$

where $\rho_{N,j}$ are the Christoffel numbers of the stand Hermite–Gauss interpolation (see, for instance, [8]). Analogously to the method in Ref. [23], we can get $\varphi^{n,\delta}$ by using Hermite-pseudospectral method with the discrete inner product and the discrete norm

$$\langle u, v \rangle_N = \sum_{j=0}^N u(\sigma_{N,j})v(\sigma_{N,j})\omega_{N,j}, \quad \|v\|_N = \langle v, v \rangle_N^{1/2}.$$

Usually it is difficult to get the analytic solution of the fractional partial differential equation (1.1) with some boundary conditions, so we have to obtain the data g by numerical methods for a given $f(x)$ at first. The numerical procedure is described as follows:

Step 1: Calculating the Hermite interpolation of f

$$(\mathcal{I}_N f)(x) = \sum_{j=0}^N \mathbf{f}_j h_j(x),$$

where $\mathbf{f}_j = \langle f, h_j \rangle_N$.

Step 2: Obtaining an approximation of the Fourier transform of g

$$\hat{g}(\xi) = \frac{1}{\lambda_\alpha(\xi)} \hat{f}(\xi) \approx \frac{1}{\lambda_\alpha(\xi)} \sum_{j=0}^{N} (-i)^j \mathbf{f}_j h_j(\xi).$$

Step 3: Computing the values

$$g(x_j) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{g}(\xi) e^{i\xi x_j} \, d\xi.$$

In the following, we present numerical results of two examples to check the effectiveness of the method. The accuracy of results are measured by

$$E_f = \left\| f^{n,\delta} - f \right\|_N = \left[\sum_{j=0}^{N} \omega_{N,j} \left(f^{n,\delta}(x_j) - f(x_j) \right)^2 \right]^{1/2}.$$

For simplicity, all examples are computed by using `Matlab` with $N = 256$, $\kappa_\alpha = 1$, $p = q = 0.5$ and $T = 1$ (If we take N, κ_α, p, q and T with other values, we can also obtain a satisfactory result).

$$E_f = \left\| f^{n,\delta} - f \right\|_N = \left[\sum_{j=0}^{N} \omega_{N,j} \left(f^{n,\delta}(x_j) - f(x_j) \right)^2 \right]^{1/2}.$$

Example 1. [18] First we consider the function

$$f(x) = \left(\frac{x^3}{4} - \frac{3x}{2} \right) e^{-\frac{x^2}{4}}.$$

Fig. 1 shows the error of the exact solution f and the regularization solution $f^{n,\delta}$ for $\alpha = 1.1, 1.5, 1.9$.

Figs. 2–3 exhibit the variation of E_f with the changes of α and δ_1, respectively.

Table 1. E_f for various α and δ.

δ_1	$\alpha = 1.1$	$\alpha = 1.5$	$\alpha = 1.9$
1e-1	0.0181	0.0366	0.0537
1e-2	0.0022	0.0072	0.0131
1e-3	2.41e-4	0.0014	0.0023
1e-4	4.04e-5	2.36e-4	3.91e-4

Table 1 lists error for different noise level δ_1. We can see that when δ_1 decreases from 0.1 to 0.0001, the relative errors become smaller.
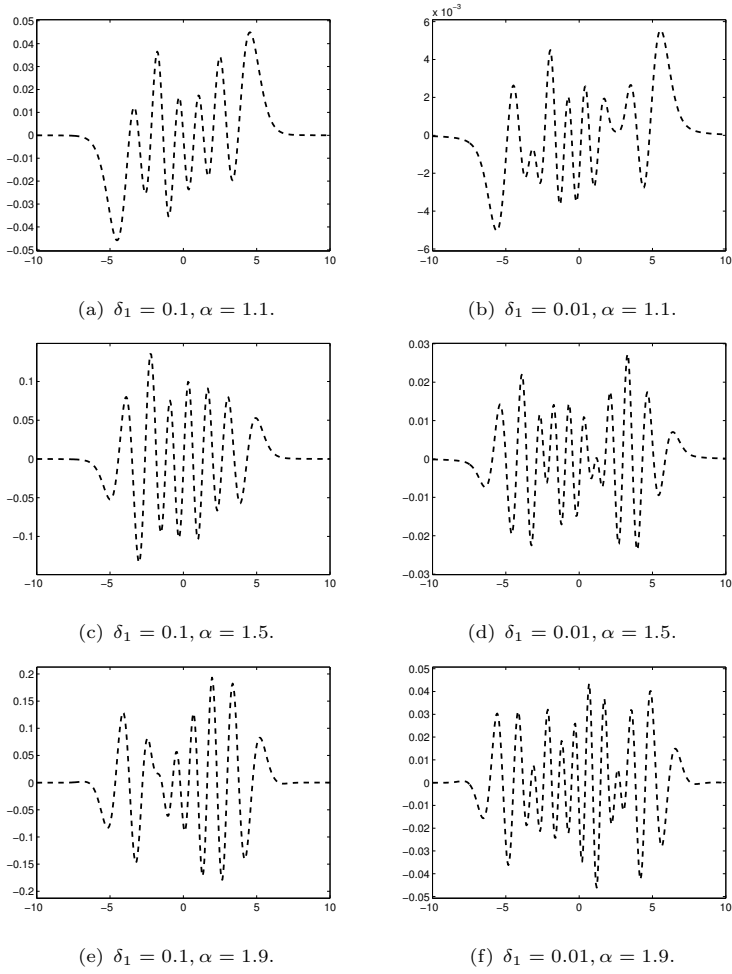
(a) $\delta_1 = 0.1, \alpha = 1.1$.

(b) $\delta_1 = 0.01, \alpha = 1.1$.

(c) $\delta_1 = 0.1, \alpha = 1.5$.

(d) $\delta_1 = 0.01, \alpha = 1.5$.

(e) $\delta_1 = 0.1, \alpha = 1.9$.

(f) $\delta_1 = 0.01, \alpha = 1.9$.

Figure 1. The error of the exact solution and its approximation (Example 1).

(a) $\delta_1 = 0.1$

(b) $\delta_1 = 0.01$
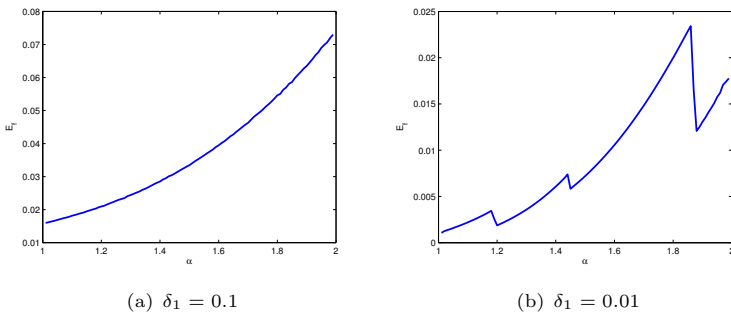
Figure 2. The variation of E_f with α (Example 1).
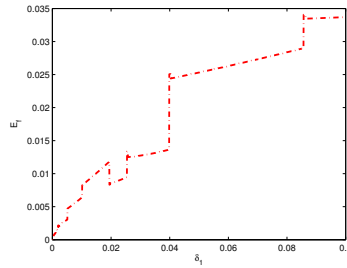
(a) $\alpha = 1.5$

Figure 3. The variation of E_f with δ_1 (Example 1).

Example 2. Consider a non-smooth

$$f(x) = \begin{cases} x + 8, & -8 \le x \le -4, \\ -x, & -4 < x \le 4, \\ x - 8, & 4 < x \le 8, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

The errors are given in Table 2 and the error of the exact and computed functions are shown in Fig. 4.

Table 2. E_f for various α and δ.

δ_1	$\alpha = 1.1$	$\alpha = 1.5$	$\alpha = 1.9$
1e-1	0.0271	0.1125	0.3106
1e-2	0.0066	0.0278	0.0674
1e-3	0.0026	0.0063	0.0179
1e-4	0.0016	0.0027	0.0092

The variation of E_f with the changes of α and δ_1 are exhibited in Figs. 5–6, respectively. It is easy to see that the numerical effect of the method is satisfactory. The numbers in Table 2 show that the convergence rate is slowly than the one in Table 1, this accords with our theoretical results.

These examples demonstrate the potential effectiveness of our approximation scheme, for suitable problems.

5 Conclusion

We have proposed an effective numerical method for determining unknown source in a space fractional diffusion equation. We have proved that the numerical method is stable and an error estimate has been obtained. Numerical tests indicate that the proposed method works well.
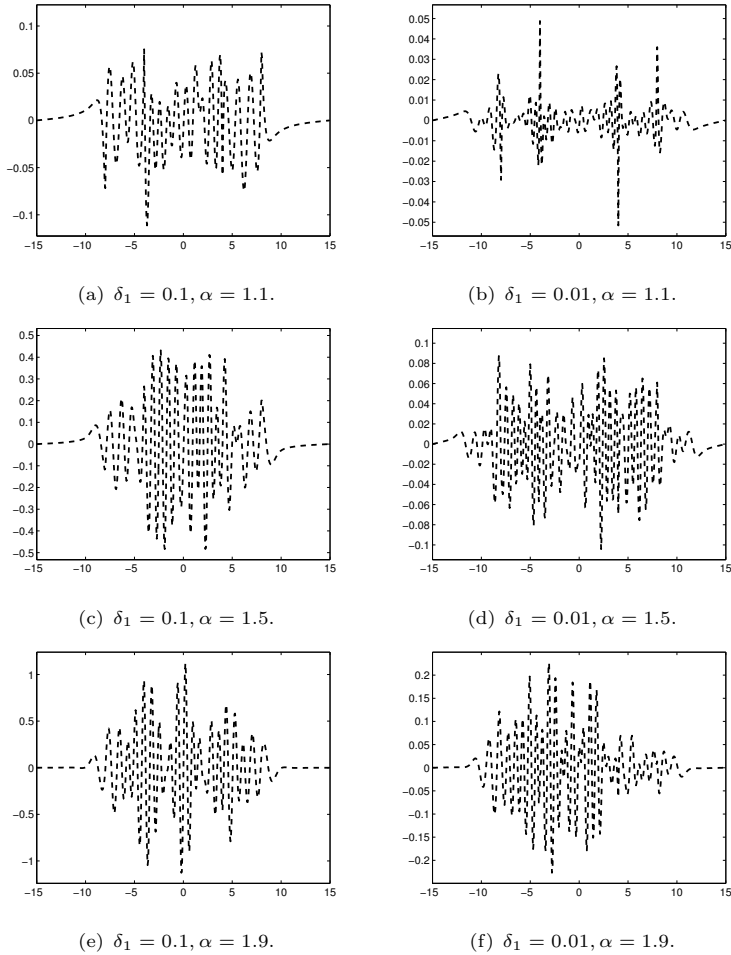
(a) $\delta_1 = 0.1, \alpha = 1.1$.

(b) $\delta_1 = 0.01, \alpha = 1.1$.

(c) $\delta_1 = 0.1, \alpha = 1.5$.

(d) $\delta_1 = 0.01, \alpha = 1.5$.

(e) $\delta_1 = 0.1, \alpha = 1.9$.

(f) $\delta_1 = 0.01, \alpha = 1.9$.

Figure 4. The error of the exact solution and its approximation (Example 2).

Acknowledgments

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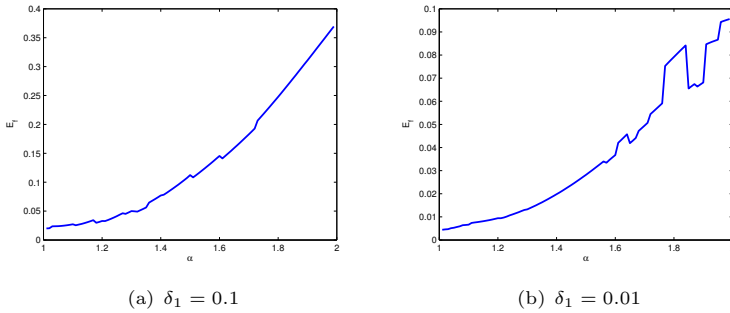
(a) $\delta_1 = 0.1$ (b) $\delta_1 = 0.01$

Figure 5. The variation of E_f with α (Example 2).
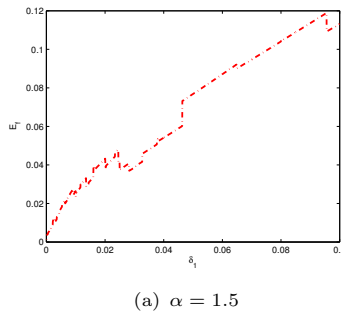

(a) $\alpha = 1.5$

Figure 6. The variation of E_f with δ_1 (Example 2).

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