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Multilinear Weighted Estimates and Quantum Zakharov System

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Abstract. We consider the well-posedness theory of the compact case of one-dimensional quantum Zakharov system with the periodic boundary condition. The global well-posedness for sufficiently regular data is shown. The semi-classical limit as $\epsilon \to 0$ is obtained on a compact time interval whereas the quantum perturbation proves to be singular on an infinite time interval.

Keywords: quantum Zakharov system, well-posedness, higher order perturbation.

AMS Subject Classification: 35B30; 35Q40; 35Q55.

1 Introduction

Consider the quantum Zakharov system (QZS)

$$\begin{cases} (i\partial_t + \alpha\partial_{xx} - \epsilon^2\partial_{xx}^2)u = un, \ (x,t) \in \mathbb{T} \times [0,T], \\ (\beta^{-2}\partial_{tt} - \partial_{xx} + \epsilon^2\partial_{xx}^2)n = \partial_{xx}(|u|^2), \\ (u(x,0), n(x,0), \partial_t n(x,0)) = (u_0, n_0, n_1) \in H^{s,l} := H^s(\mathbb{T}) \times H^l(\mathbb{T}) \times H^{l-2}(\mathbb{T}), \end{cases} \tag{1.1}$$

where $u(x,t) \in \mathbb{C}$, $n(x,t) \in \mathbb{R}$, $T > 0$, $\alpha, \beta > 0$, $s, l \in \mathbb{R}$. The two conserved quantities that are of use in this paper are

$$M[u,n,\partial_t n](t) = \|u\|_{L^2}^2 = \|u_0\|_{L^2}^2, \quad H[u,n,\partial_t n](t) = \alpha\|\partial_x u\|_{L^2}^2 + \epsilon^2\|\partial_{xx}u\|_{L^2}^2 + \frac{1}{2}\Big(\|n\|_{L^2}^2 + \frac{1}{\beta^2}\|\partial_t n\|_{\dot{H}^{-1}}^2 + \epsilon^2\|\partial_x n\|_{L^2}^2\Big) + \int n|u|^2. \tag{1.2}$$

When $\epsilon = 0$, QZS is well-known as the classical Zakharov system (ZS) that models the interaction of Langmuir turbulence waves and ion-acoustic waves.

Here u denotes the slowly-varying envelope of electric field, and n represents an ion-acoustic wave that models the density fluctuation of ions [17]. A thrust of interest in studying the quantum effects unexplained by ZS came from the physics community [6]. There the quantum effect is characterized by a fourth-order derivative perturbation with a quantum parameter $\epsilon > 0$ that is non-negligible when either the ion-plasma frequency is high or the electrons' temperature is low; for more background in the physics of this model, see [9, 14].

Although the QZS model is relatively new, the Bourgain norm method has been used successfully by many including, but not limited to, [1, 7, 11, 12], in applications to various dispersive equations such as the KdV, nonlinear Schrödinger equation, and ZS. The task of proving boundedness for certain multilinear operators reduces to showing spacetime Lebesgue-type estimates in the Fourier space, which can be a challenge on bounded domains where satisfactory Strichartz estimates are not available. Despite this difficulty, see [2, 4, 13] for the well-posedness theory of ZS on periodic domains. For more recent work on QZS on $\mathbb{R}$, see [3, 5, 10].

Our goal is to understand the effect of quantum modification represented by the additional biharmonic operator on $\mathbb{T}$, thereby extending the results of [15]. We show that the biharmonic operator provides an extra degree of smoothing that nullifies the distinction between resonance ($\frac{\beta}{\alpha} \in \mathbb{Z}$) and non-resonance ($\frac{\beta}{\alpha} \notin \mathbb{Z}$), a phenomenon that played a central role in [15]. More precisely, we show that the region of Sobolev exponent pairs $(s, l) \in \mathbb{R}^2$ yielding well-posedness for $\epsilon = 0$ (which depend on $\frac{\beta}{\alpha} \in \mathbb{Z}$ or $\frac{\beta}{\alpha} \notin \mathbb{Z}$) are no longer different when $\epsilon > 0$. It is shown that if ZS is well-posed with data in certain Sobolev spaces, then so is QZS. Under the condition $s \geq 0$, it is shown that our method yields a region of Sobolev exponents for the local well-posedness that is sharp up to the boundary.

For $s, l \in \mathbb{R}$, $b \in \mathbb{R}$, define

$$\begin{aligned} \|f\|_{X_S^{s,b}} &= \|\langle k\rangle^s \langle \tau + \alpha k^2 + \epsilon^2 k^4\rangle^b \widehat{f}(k,\tau)\|_{L^2_\tau l^2_k}, \\ \|f\|_{X_W^{l,b}} &= \|\langle k\rangle^l \langle |\tau| - \beta|k|\langle \epsilon k\rangle\rangle^b \widehat{f}(k,\tau)\|_{L^2_\tau l^2_k}, \end{aligned} \tag{1.3}$$

and the augmented norm

$$\begin{aligned} \|f\|_{Y_S^s} &= \|f\|_{X_S^{s,\frac{1}{2}}} + \|\widehat{f}(k,\tau)\langle k\rangle^s\|_{l^2_k L^1_\tau}, \\ \|f\|_{Y_W^l} &= \|f\|_{X_W^{l,\frac{1}{2}}} + \|\widehat{f}(k,\tau)\langle k\rangle^s\|_{l^2_k L^1_\tau}, \end{aligned} \tag{1.4}$$

where k, τ denote the Fourier dual variables to the physical variables x, t, and $\widehat{f}$ (or $\mathcal{F}f$) denotes either the space or spacetime Fourier transform, depending on the context. Let the restricted space be

$$\|f\|_{X_{S,T}^{s,b}} = \inf_{\tilde{f}=f,\, t\in[0,T]} \|\tilde{f}\|_{X_S^{s,b}},$$

and similarly for norms defined in (1.3) and (1.4). Further define the region of well-posedness

$$\begin{aligned} \Omega_L &= \{s \geq 0,\ -1 \leq l < 2s+1,\ -2 < s-l \leq 2\}, \\ \Omega_G &= \{0 \leq s-l \leq 2,\ s+l \geq 4\} \cup \{(2,1)\}. \end{aligned}$$

Theorem 1. *If $(s,l) \in \Omega_L$, then (1.1) is locally well-posed; there exists $T = T(\|(u_0,n_0,n_1)\|_{H^{s,l}}) > 0$ and a unique $(u,n,\partial_t n) \in Y^s_{S,T} \times Y^l_{W,T} \times Y^{l-2}_{W,T}$ that satisfies (1.1). Further, if $T' \in (0,T)$, then there exists a neighborhood $B \subseteq H^{s,l}$ around (u_0,n_0,n_1) such that the data-to-solution map $(u_0,n_0,n_1) \mapsto (u,n,\partial_t n)$ is Lipschitz-continuous from B to $Y^s_{S,T} \times Y^l_{W,T} \times Y^{l-2}_{W,T}$.*

In the Appendix, explicit examples are given that illustrate the necessity of

$$s \geq -1,\ -1 \leq l \leq 2s+1,\ -2 \leq s-l \leq 2.$$

Theorem 2. *If $(s,l) \in \Omega_G$, then the unique local solution obtained in Theorem 1 can be extended globally in time. More precisely, there exists $(u,n,\partial_t n) \in C_{loc}([0,\infty); H^{s,l})$ that satisfies (1.1) such that for all $T > 0$, $(u,n,\partial_t n)$ is a unique solution in $Y^s_{S,T} \times Y^l_{W,T} \times Y^{l-2}_{W,T}$.*

Lastly the semi-classical limit of QZS to ZS as $\epsilon \to 0$ is considered. We extend the results of [8] to show that the solutions behave continuously as $\epsilon \to 0$ on a compact time interval. On the other hand, explicit examples are given to illustrate that the biharmonic operator $\epsilon^2\Delta^2$, for any $\epsilon > 0$, is a singular perturbation on an infinite time interval. Here we address a subtlety that QZS generates flow on $H^{s,l}$ whereas the classical ZS does so on $H^{s,l}_0 := H^s(\mathbb{T}) \times H^l(\mathbb{T}) \times H^{l-1}(\mathbb{T})$. To overcome this apparent gap of solution space, the growth of Sobolev norm of solutions is estimated in various function spaces with bounds independent of $\epsilon > 0$.

We outline the organization of the paper. In Section 2, notations are introduced. In Section 3, we summarize a set of linear estimates that are used throughout the paper. In Section 4, nonlinear estimates are proved and applied to yield local well-posedness of (1.1). In Section 5, the local solutions obtained in Section 4 are extended to global solutions for every $\epsilon > 0$, and the limit $\epsilon \to 0$ is considered.

2 Notation

Given $A_\pm$, denote $\sum_\pm A_\pm := A_+ + A_-$. Let $\psi \in C_c^\infty(\mathbb{R})$ be a smooth cutoff function with a compact support in $[-2,2]$ and $\psi(t) = 1$ for all $t \in [-1,1]$. For $b \in \mathbb{R}$, we write $b\pm$ to denote $b \pm \epsilon'$ for some universal $\epsilon' \ll 1$. Assume $\epsilon \leq 1$. Let $\langle k \rangle = (1+k^2)^{\frac{1}{2}}$ and define $\langle \nabla \rangle$, $D := |\nabla|$ as multipliers.

Assume without loss of generality that n_0, n_1 have zero means. If $\int n_0 dx$, $\int n_1 dx \neq 0$, then consider the change of variable

$$u(x,t) \mapsto e^{i\left(\frac{t^2}{4\pi}\int n_1 + \frac{t}{2\pi}\int n_0\right)} u(x,t),\ n(x,t) \mapsto n(x,t) - \frac{t}{2\pi}\int n_1 - \frac{1}{2\pi}\int n_0,$$

which, by direct computation, satisfies (1.1) with zero means in the new variable. By integrating the second equation of (1.1) over space, one obtains $\frac{d^2}{dt^2}\int_{\mathbb{T}} n = 0$, and, therefore, the mean zero condition on n_0, n_1 allows us to make sense of $\|\partial_t n\|_{\dot{H}^{-1}}$ in (1.2).

The contraction mapping argument is developed to show

$$u(t) = \Gamma_1(u,n)(t) := U_\epsilon(t)u_0 - i\int_0^t U_\epsilon(t-t')(un)(t')dt',$$
$$n(t) = \Gamma_2(u)(t) := \partial_t V_\epsilon(t)n_0 + V_\epsilon(t)n_1 + \beta^2 \int_0^t V_\epsilon(t-t')\partial_{xx}(|u|^2)(t')dt', \qquad (2.1)$$

where $U_\epsilon(t), V_\epsilon(t), \partial_t V_\epsilon(t)$ for $\epsilon \geq 0$ are defined via Fourier multipliers

$$e^{-it(\alpha k^2+\epsilon^2 k^4)}, \quad \frac{\sin(\beta|k|\langle \epsilon k\rangle t)}{\beta|k|\langle \epsilon k\rangle}, \quad \cos(\beta|k|\langle\epsilon k\rangle t),$$

respectively on $\mathbb{Z}$.

To control the nonlinear contribution coming from the Duhamel term of (2.1), consider the companion spaces to Y^s_S, Y^l_W:

$$\|f\|_{Z^s_S} = \|f\|_{X_S^{s,-\frac{1}{2}}} + \left\| \frac{\langle k\rangle^s}{\langle \tau + \alpha k^2 + \epsilon^2 k^4\rangle} \widehat{f}(k,\tau)\right\|_{l^2_k L^1_\tau},$$
$$\|f\|_{Z^l_W} = \|f\|_{X_W^{l,-\frac{1}{2}}} + \left\| \frac{\langle k\rangle^l}{\langle |\tau| - \beta|k|\langle \epsilon k\rangle\rangle} \widehat{f}(k,\tau)\right\|_{l^2_k L^1_\tau}.$$

3 Linear estimates

Fix $\alpha, \beta, \epsilon \geq 0$ and $T \in (0,1]$ for Sections 3 and 4.

Lemma 1 [Homogeneous estimates]. *For $s, l \in \mathbb{R}$,*

$$\|U_\epsilon(t)u_0\|_{H^s} = \|u_0\|_{H^s}, \ \|\psi(t)U_\epsilon(t)u_0\|_{Y^s_S} \lesssim_\psi \|u_0\|_{H^s}, \forall \epsilon \geq 0.$$
$$\|\partial_t V_\epsilon(t)n_0\|_{H^l} \leq \|n_0\|_{H^l}, \ \|\psi(t)\partial_t V_\epsilon(t)n_0\|_{Y^l_W} \lesssim_\psi \|n_0\|_{H^l}, \forall\epsilon \geq 0.$$
$$\|V_\epsilon(t)n_1\|_{H^l} \lesssim (t + \frac{1}{\beta\epsilon})\|n_1\|_{H^{l-2}}, \ \|\psi(t)V_\epsilon(t)n_1\|_{Y^l_W} \lesssim_\psi (1+\frac{1}{\beta\epsilon})\|n_1\|_{H^{l-2}},$$
$$\|V_0(t)n_1\|_{H^l} \lesssim (t+\frac{1}{\beta})\|n_1\|_{H^{l-1}}, \ \|\psi(t)V_0(t)n_1\|_{Y^l_W} \lesssim_\psi (1+\frac{1}{\beta})\|n_1\|_{H^{l-1}}.$$

Proof. The first line of inequalities follows from the unitarity of Schrödinger operator; see [16, Lemma 2.8]. A similar argument can be used to show the other inequalities. □

Lemma 2 [Inhomogeneous estimates]. *For $s, l \in \mathbb{R}$ and $\rho \in [0,1]$,*

$$\left\|\psi(t)\int_0^t U_\epsilon(t-t')F(t')dt'\right\|_{Y^s_S} \lesssim_\psi \|F\|_{Z^s_S},$$
$$\left\|\psi(t)\int_0^t V_\epsilon(t-t')D^{2-\rho}F(t')dt'\right\|_{Y^l_W} \lesssim_\psi c(\rho,\beta,\epsilon)\|F\|_{Z^l_W},$$
$$\left\|\psi(t)\int_0^t \partial_t V_\epsilon(t-t')D^{2-\rho}F(t')dt'\right\|_{Y^{l-2}_W} \lesssim_\psi \|F\|_{Z^l_W}.$$

Proof. The first inequality is standard in literature; see [16, Proposition 2.12]. The second and third are proved similarly where
$c(\rho,\beta,\epsilon) = \left(\frac{1-\rho}{\rho\epsilon^2}\right)^{\frac{1-\rho}{2}}/\beta\rho^{-1/2}$. □

Lemma 3. *Let $T \le 1$, $s, l \in \mathbb{R}$, and $-\frac{1}{2} < b \le b' < \frac{1}{2}$. Then*

$$\|\psi(t/T)u\|_{X_S^{s,b}} \lesssim_{\psi,b,b'} T^{b'-b}\|u\|_{X_S^{s,b'}},\ \|\psi(t/T)u\|_{X_W^{l,b}} \lesssim_{\psi,b,b'} T^{b'-b}\|u\|_{X_W^{l,b'}}.$$

Proof. The first inequality follows from [16, Lemma 2.11]. The second inequality follows similarly. □

4 Nonlinear estimates

Proposition 1. *For $0 < \rho \le 1$, suppose $s \ge 0$, $-1 \le l \le 2s + 1 - \rho$, $-2 + \rho \le s - l \le 2$ and $b \in (\frac{1}{6}, \frac{1}{2}]$. Then there exists $C = C(\alpha, \beta, \epsilon, \rho, s, l, b) > 0$ such that*

$$\|un\|_{X_S^{s,-\frac{1}{2}}} \le C(\|u\|_{X_S^{s,b}}\|n\|_{X_W^{l,\frac{1}{2}}} + \|u\|_{X_S^{s,\frac{1}{2}}}\|n\|_{X_W^{l,b}}),$$
$$\|D^\rho(u\overline{v})\|_{X_W^{l,-\frac{1}{2}}} \le C(\|u\|_{X_S^{s,b}}\|v\|_{X_S^{s,\frac{1}{2}}} + \|u\|_{X_S^{s,\frac{1}{2}}}\|v\|_{X_S^{s,b}}).$$

Proposition 2. *Assuming the hypotheses of Proposition 1,*

$$\left\|\frac{\langle k\rangle^s}{\langle \tau + \alpha k^2 + \epsilon^2 k^4\rangle}\widehat{un}(k,\tau)\right\|_{l_k^2 L_\tau^1} \lesssim \|u\|_{X_S^{s,b}}\|n\|_{X_W^{l,\frac{1}{2}}} + \|u\|_{X_S^{s,\frac{1}{2}}}\|n\|_{X_W^{l,b}},$$
$$\left\|\frac{\langle k\rangle^l}{\langle |\tau| - \beta|k|\langle \epsilon k\rangle\rangle}\widehat{D^\rho(u\overline{v})}(k,\tau)\right\|_{l_k^2 L_\tau^1} \lesssim \|u\|_{X_S^{s,b}}\|v\|_{X_S^{s,\frac{1}{2}}} + \|u\|_{X_S^{s,\frac{1}{2}}}\|v\|_{X_S^{s,b}}.$$

The contraction mapping argument can be closed by combining Propositions 1 and 2 and Lemma 3.

Corollary 1. For some $\theta \in (0, \frac{1}{3})$,

$$\|un\|_{Z_S^s} \lesssim T^\theta\|u\|_{Y_S^s}\|n\|_{Y_W^l},\ \|D^\rho(|u|^2)\|_{Z_W^l} \lesssim T^\theta\|u\|_{Y_S^s}^2.$$

Remark 1. The proof of Proposition 2 is similar to that of Proposition 1, and thus is omitted. See [15] for more details.

Remark 2. The method of direct estimation by the Cauchy-Schwarz inequality does not seem to work, at least directly, when $\rho = 0$. One can check that the τ_1-integral in (4.7) is not justified. In fact if $k = 0$, then $IV = \infty$ by direct computation.

Assuming the aforementioned statements, we prove Theorem 1.

Proof. For $T > 0$, define $X = Y_{S,T}^s \times Y_{W,T}^l \times Y_{W,T}^{l-2}$ with $\|(u, n, \partial_t n)\| := \|u\|_{X_{S,T}^s}$ $+\|n\|_{Y_{W,T}^l} + \|\partial_t n\|_{Y_{W,T}^{l-2}}$, $X(R) = \{(u, n, \partial_t n) \in X : \|(u, n, \partial_t n)\| \le R\sigma\}$ for $R > 0$ to be determined and $\sigma = \|(u_0, n_0, n_1)\|_{H^{s,l}}$.

Consider $\Gamma(u,n,\partial_t n) = (\Gamma_1(u,n), \Gamma_2(u), \Gamma_3(u))$ on $X(R)$ where Γ_1, Γ_2 are defined in (2.1) and $\Gamma_3(u)(t) = \partial_t \Gamma_2(u)(t)$. If $(s,l) \in \Omega_L$, pick $\rho > 0$ sufficiently small such that the hypotheses of Proposition 1 are fulfilled. By Corollary 1, there exists $\theta > 0$ such that

$$\|\Gamma(u,n,\partial_t n)\| \lesssim \sigma + T^\theta \|u\|_{Y^s_{S,T}} \|n\|_{Y^l_{W,T}} + T^\theta \|u\|^2_{Y^s_{S,T}} \leq \sigma + 2T^\theta R^2 \sigma^2.$$

If R is chosen sufficiently big (depending on the given parameters), then by choosing $0 < T \lesssim \sigma^{-\frac{1}{\theta}}$, Γ maps into $X(R)$. Similarly given $(u_1, n_1, \partial_t n_1), (u_2, n_2, \partial_t n_2) \in X$,

$$\|\Gamma(u_1,n_1,\partial_t n_1) - \Gamma(u_2,n_2,\partial_t n_2)\| \lesssim T^\theta R\sigma \|(u_1,n_1,\partial_t n_1) - (u_2,n_2,\partial_t n_2)\|,$$

and by choosing $0 < T \lesssim \sigma^{-\frac{1}{\theta}}$ sufficiently small, Γ is a contraction on $X(R)$ and hence there exists a unique $(u,n,\partial_t n) \in X \hookrightarrow C([0,T]; H^{s,l})$ that satisfies (1.1). The local Lipschitz continuity of the solution map immediately follows from the contraction mapping argument. □

The goal is to show the boundedness of the multilinear operators corresponding to the nonlinear terms which, at a technical level, involves directly estimating a $L^\infty L^1$-norm of a function defined on the spacetime Fourier space in different regions depending on which dispersive weight is most dominant. Observing that

$$\begin{aligned}(\tau + \alpha k^2 + \epsilon^2 k^4) &- (\tau_1 + \alpha k_1^2 + \epsilon^2 k_1^4) - (\tau_2 \pm \beta k_2 \langle \epsilon k_2 \rangle) \\ &= (k - k_1)\Big((k + k_1)(\alpha + \epsilon^2(k^2 + k_1^2)) \mp \beta \langle \epsilon (k - k_1) \rangle\Big),\end{aligned}$$

we obtain

$$\begin{aligned}\max\Big(|\tau + \alpha k^2 + \epsilon^2 k^4|, &|\tau_1 + \alpha k_1^2 + \epsilon^2 k_1^4|, ||\tau_2| - \beta |k_2| \langle \epsilon k_2 \rangle|\Big) \\ &\geq \frac{1}{3}|k - k_1| \left|(k + k_1)(\alpha + \epsilon^2(k^2 + k_1^2)) \mp \beta \langle \epsilon (k - k_1) \rangle\right|,\end{aligned} \tag{4.1}$$

where the sign on the RHS of (4.1) depends on τ_2, k_2. Since this subtlety does not affect our subsequent analysis, we do not keep track of the sign. For the sake of notational convenience, define

$$h(k,k_1) = (k + k_1)(\alpha + \epsilon^2(k^2 + k_1^2)) \mp \beta \langle \epsilon (k - k_1) \rangle.$$

Lemma 4. *[4, Lemma 3.3] If $\delta \geq \gamma \geq 0$ and $\delta + \gamma > 1$, then*

$$\int \frac{d\tau}{\langle \tau - a_1 \rangle^\delta \langle \tau - a_2 \rangle^\gamma} \lesssim \langle a_1 - a_2 \rangle^{-\gamma} \phi_\delta(a_1 - a_2), \phi_\delta(a) \simeq \begin{cases} 1, & \delta > 1, \\ \log(1 + \langle a \rangle), & \delta = 1, \\ \langle a \rangle^{1-\delta}, & \delta < 1. \end{cases}$$

Lemma 5. *For all* $e_1 > \frac{1}{4}, e_2 > \frac{1}{3}$,

$$\sigma_1(k,\tau) := \sum_{k_1 \neq k; \pm} \frac{1}{\langle \epsilon^2 k_1^4 + \alpha k_1^2 + \tau \pm \beta(k-k_1)\langle \epsilon(k-k_1)\rangle\rangle^{e_1}} \leq c_1(\alpha,\beta,\epsilon,e_1) < \infty,$$

$$\sigma_2(k,\tau) := \sum_{k_1} \frac{1}{\langle k_1^3 - \frac{3k}{2}k_1^2 + \left(\frac{\alpha+2\epsilon^2 k^2}{2\epsilon^2}\right)k_1 + \frac{\tau - \alpha k^2 - \epsilon^2 k^4}{4\epsilon^2 k}\rangle^{e_2}} \leq c_2(\alpha,\beta,\epsilon,e_2) < \infty, \text{ if } k \neq 0.$$

Proof. The second inequality can be proved similarly as [4, lemma 3(c)]. For the first inequality, there exists $c > 0$ independent of k, k_1 such that

$$\big|(k-k_1)\langle\epsilon(k-k_1)\rangle - (k-k_1)|\epsilon(k-k_1)|\,\big| \leq c.$$

Hence the term $\langle\epsilon(k-k_1)\rangle$ in the summation can be replaced with $|\epsilon(k-k_1)|$. Then

$$\sigma_1(k,\tau) \leq \sum_{k_1\neq k;\pm} \frac{1}{\langle \epsilon^2 k_1^4 + \alpha k_1^2 + \tau \pm \beta\epsilon(k-k_1)^2\rangle^{e_1}} \leq c',$$

where the constant is independent of k, τ by an argument similar to [4, lemma 3(c)]. □

Lemma 6. *There exist* $C(\alpha,\beta,\epsilon), c(\alpha,\beta,\epsilon) > 0$ *such that for all* $(k,k_1) \in \mathbb{Z}^2$ *that satisfies* $\{|k| \geq C(\alpha,\beta,\epsilon)\} \cup \{|k_1| \geq C(\alpha,\beta,\epsilon)\}$, *the estimate,* $|h(k,k_1)| \geq c(\alpha,\beta,\epsilon)|k-k_1|$, *holds.*

Proof. Assume $k \neq k_1$. For a fixed $k \in \mathbb{Z}$, let $r_\mp(k) \in \mathbb{R}$ be the unique real-root of $h(k,\cdot)$, where $r_-(k)$ corresponds to the minus sign in $h(k,\cdot)$, and similarly for $r_+(k)$; we drop the $\mp$-subscript. Noting that h is symmetric in both arguments, it suffices to assume $|k| \geq C(\alpha,\beta,\epsilon)$, where

$$C(\alpha,\beta,\epsilon) := \max\left(C_1(\alpha,\beta,\epsilon), \sqrt{\frac{3\sqrt{2}\beta}{\epsilon}}, \frac{1}{3\epsilon}\right), \tag{4.2}$$

where for all $|k| \geq C_1(\alpha,\beta,\epsilon) > 0$, we have $\beta\langle\epsilon k\rangle < |k|(\alpha + \epsilon^2 k^2)$.

We first show that for k sufficiently big, $r(k) \notin \mathbb{Z}$. For $k \in \mathbb{Z}$, consider the graphs of $k_1 \mapsto (k+k_1)(\alpha + \epsilon^2(k^2+k_1^2))$ and $k_1 \mapsto \pm\beta\langle\epsilon(k-k_1)\rangle$. If the y-intercept of the cubic polynomial is greater (in magnitude) than that of the square-root term, i.e., $\beta\langle\epsilon k\rangle < |k|(\alpha+\epsilon^2k^2)$, then $r(k) \in [-c_2 k, 0]$ for $k > 0$ and $r(k) \in [0, -c_2 k]$ for $k < 0$, where $c_2 = c_2(\alpha,\beta,\epsilon) > 0$.

Now we claim $\lim_{|k|\to\infty} |r(k)+k| = 0$. From $h(k,r(k)) = 0$,

$$|r(k)+k| = \left|\frac{\beta\langle\epsilon(k-r(k))\rangle}{\alpha+\epsilon^2(k^2+r(k)^2)}\right| \lesssim \frac{\beta\epsilon|k-r(k)|}{\alpha+\epsilon^2k^2} \leq \frac{(1+c_2)\beta\epsilon|k|}{\alpha+\epsilon^2k^2} \xrightarrow[|k|\to\infty]{} 0.$$

Hence, if $|k|$ is sufficiently big and $r(k) \in \mathbb{Z}$, then $r(k) = -k$, which cannot be since $|h(k,-k)| = \beta\langle 2\epsilon k\rangle \geq \beta$. For $k \in \mathbb{Z}$, to show $\inf_{k_1\in\mathbb{Z}} |h(k,k_1)|$ is attained at $k_1 = -k$, note that from standard calculus,

$$\partial_{k_1} h(k,k_1) = 3\epsilon^2 k_1^2 + 2\epsilon^2 k k_1 + \alpha + \epsilon^2 k^2 \pm \frac{\beta\epsilon^2(k-k_1)}{\langle\epsilon(k-k_1)\rangle} \geq \alpha + \frac{2}{3}\epsilon^2 k^2 \pm \frac{\beta\epsilon^2(k-k_1)}{\langle\epsilon(k-k_1)\rangle},$$

and since $\frac{\beta\epsilon^2|k-k_1|}{\langle\epsilon(k-k_1)\rangle} \leq \beta\epsilon$, it follows that $\partial_{k_1} h \geq \alpha$ by (4.2), and hence

$$\inf_{|k|\geq C(\alpha,\beta,\epsilon),\ (k,k_1)\in\mathbb{Z}^2} |h(k,k_1)| \geq \beta.$$

If $|k-k_1| \leq 3|k|$, then $\inf_{|k|\geq C(\alpha,\beta,\epsilon),\ (k,k_1)\in\mathbb{Z}^2} |\frac{h(k,k_1)}{k-k_1}| \geq \frac{\beta}{3C(\alpha,\beta,\epsilon)}$. If $|k-k_1| \geq 3|k|$, then $|k_1| \geq 2|k|$, $|k+k_1| \geq \frac{|k_1|}{2}$, and $|k-k_1| \leq \frac{3|k_1|}{2}$. Furthermore,

$$\left|\frac{h(k,k_1)}{k-k_1}\right| \geq \frac{1}{3}(\alpha+\epsilon^2(k^2+k_1^2)) - \left|\beta\frac{\langle\epsilon(k-k_1)\rangle}{k-k_1}\right| \geq \frac{1}{3}(\alpha+\epsilon^2(k^2+k_1^2)) - \sqrt{2}\beta\epsilon \geq \frac{\alpha}{3},$$

where the last inequality is by (4.2). □

Lemma 7. *There exists $C(\alpha,\beta,\epsilon) > 0$ such that if $\{0 \neq |k| \geq 2|k_1|\} \cap \{|k| \geq C(\alpha,\beta,\epsilon)\}$, then $|k-k_1||h(k,k_1)| \gtrsim |k|^4$. Similarly if $\left\{0 \neq \frac{|k_1|}{2} \geq |k|\right\} \cap \{|k_1| \geq C(\alpha,\beta,\epsilon)\}$, then $|k-k_1||h(k,k_1)| \gtrsim |k_1|^4$.*

Proof. Since h is symmetric in k, k_1, it suffices to prove the first statement. If $|k| \geq 2|k_1|$, then $|k \pm k_1| \geq \frac{|k|}{2}$. If we further assume $|k| \geq \frac{2}{\epsilon}$, then $\epsilon|k-k_1| \geq \frac{\epsilon}{2}|k| \geq 1$, and therefore,

$$\beta\langle\epsilon(k-k_1)\rangle \leq \sqrt{2}\beta\epsilon|k-k_1| \leq \sqrt{2}\beta\epsilon(|k|+|k_1|) \leq \frac{3}{\sqrt{2}}\beta\epsilon|k|.$$

By the triangle inequality, if $|k| \geq 10\max(1/\epsilon, \sqrt{\beta/\epsilon})$,

$$|k-k_1||h(k,k_1)| \geq \frac{|k|}{2}\Big(\frac{\epsilon^2}{2}|k|^3 - \frac{3}{\sqrt{2}}\beta\epsilon|k|\Big) \geq \frac{\epsilon^2}{8}|k|^4.$$

□

The proof of Proposition 1 is given below.

Proof. Though the main idea of this proof follows closely that of [15], we include a full proof here to address any subtleties that rise from the fourth-order perturbation. To use the duality argument, let $w \in L^2_{k,\tau}$, $\|w\|_{L^2} = 1$ and $w \geq 0$. Since

$$\|un\|_{X_S^{s,-\frac{1}{2}}} = \left\|\frac{\langle k\rangle^s}{\langle\tau+\alpha k^2+\epsilon^2 k^4\rangle^{1/2}} \sum_{k_1+k_2=k} \int_{\tau_1+\tau_2=\tau} \hat{u}(\tau_1,k_1)\hat{n}(\tau_2,k_2)\right\|_{l^2_k L^2_\tau},$$

it suffices to estimate

$$\sum_{k_1+k_2-k=0} \int_{\tau_1+\tau_2-\tau=0} \frac{\langle k\rangle^s\langle k_1\rangle^{-s}\langle k_2\rangle^{-l} f(\tau_1,k_1)g(\tau_2,k_2)w(\tau,k)}{\langle\tau+\alpha k^2+\epsilon^2k^4\rangle^{1/2}\langle\tau_1+\alpha k_1^2+\epsilon^2k_1^4\rangle^{b_1}\langle|\tau_2|-\beta|k_2|\langle\epsilon k_2\rangle\rangle^{b_2}} =: E,$$

where

$$f(\tau,k)=|\hat{u}(\tau,k)|\langle k\rangle^s\langle\tau+\alpha k^2+\epsilon^2k^4\rangle^{b_1},\ g(\tau,k)=|\hat{n}(\tau,k)|\langle k\rangle^l\langle|\tau|-\beta|k|\langle\epsilon k\rangle\rangle^{b_2},$$

and $b_1, b_2 \le \frac{1}{2}$. By direct computation, one can rule out $k_2 = 0$, and hence we assume the sum is over $k_2 \neq 0$, or equivalently, $k_1 \neq k$.

I. $\max\Big(|\tau+\alpha k^2+\epsilon^2k^4|, |\tau_1+\alpha k_1^2+\epsilon^2k_1^4|, ||\tau_2|-\beta|k_2|\langle\epsilon k_2\rangle|\Big) = |\tau+\alpha k^2+\epsilon^2k^4|.$

Let $b_1 = b_2 = b = \frac{1}{2}-$. Applying the Cauchy-Schwarz inequality in variables (k_1,τ_1) and (k_2,τ_2), followed by the Young's inequality, it suffices to show

$$\sup_{\tau,k}\frac{\langle k\rangle^{2s}}{\langle\tau+\alpha k^2+\epsilon^2k^4\rangle}\sum_{k_1\neq k}\int\frac{\langle k_1\rangle^{-2s}\langle k-k_1\rangle^{-2l}d\tau_1}{\langle\tau_1+\alpha k_1^2+\epsilon^2k_1^4\rangle^{2b}\langle|\tau-\tau_1|-\beta|k-k_1|\langle\epsilon(k-k_1)\rangle\rangle^{2b}}$$
$$=:\sup_{\tau,k} I<\infty, \tag{4.3}$$

since

$$E\lesssim\Big(\sup_{\tau,k}I\Big)^{1/2}\|u\|_{X_S^{s,b}}\|n\|_{X_W^{l,b}}\|w\|_{L^2_{k,\tau}}.$$

Let $|k|/2 \le |k_1| \le 2|k|$. Integrating in τ_1 via Lemma 4 and noting that $\langle k\rangle^{2s}\langle k_1\rangle^{-2s}\simeq 1$, we have

$$I\lesssim\sum_{k_1\neq k;\pm}\frac{1}{\langle\tau+\alpha k^2+\epsilon^2k^4\rangle\langle k-k_1\rangle^{2l}\langle\epsilon^2k_1^4+\alpha k_1^2+\tau\pm\beta(k-k_1)\langle\epsilon(k-k_1)\rangle\rangle^{4b-1}}. \tag{4.4}$$

If $|k|\lesssim 1$, then $\langle k-k_1\rangle\simeq 1$, and, therefore, the sum above is finite by Lemma 5. On the other hand, by Lemma 6, if $|k|\ge C(\alpha,\beta,\epsilon)$,

$$I\lesssim\sum_{k_1\neq k;\pm}\frac{1}{\langle k-k_1\rangle^{2l+2}\langle\epsilon^2k_1^4+\alpha k_1^2+\tau\pm\beta(k-k_1)\langle\epsilon(k-k_1)\rangle\rangle^{4b-1}}\le c_1,$$

since $l\ge -1$. Now let $|k|\ge 2|k_1|$. In this region, we have $\frac{|k|}{2}\le|k-k_1|\le\frac{3|k|}{2}$ and by Lemma 7, if $|k|\ge C(\alpha,\beta,\epsilon)$, then $\langle\tau+\alpha k^2+\epsilon^2k^4\rangle\gtrsim\langle k\rangle^4$ and

$$I\lesssim\sum_{k_1\neq k;\pm}\frac{\langle k_1\rangle^{-2s}}{\langle k\rangle^{2l-2s+4}\langle\epsilon^2k_1^4+\alpha k_1^2+\tau\pm\beta(k-k_1)\langle\epsilon(k-k_1)\rangle\rangle^{4b-1}}\le c_1,$$

since $s-l\le 2$ and $s\ge 0$. If $|k|\lesssim 1$, then by Lemma 5, $I\lesssim\sigma_1\le c_1$. Lastly if $\frac{|k_1|}{2}\ge|k|$, then $\frac{|k_1|}{2}\le|k-k_1|\le\frac{3|k_1|}{2}$ and by treating $|k_1|\ge C(\alpha,\beta,\epsilon)$ and $|k_1|\le C(\alpha,\beta,\epsilon)$ separately as above, (4.3) has been shown.

II. $\max\Big(|\tau+\alpha k^2+\epsilon^2k^4|, |\tau_1+\alpha k_1^2+\epsilon^2k_1^4|, ||\tau_2|-\beta|k_2|\langle\epsilon k_2\rangle|\Big) = |\tau_1+\alpha k_1^2+\epsilon^2k_1^4|.$

Arguing as above, it suffices to show

$$\sup_{\tau_1,k_1} II := \sup_{\tau_1,k_1}\frac{\langle k_1\rangle^{-2s}}{\langle\tau_1+\alpha k_1^2+\epsilon^2k_1^4\rangle}$$
$$\times\sum_{k\neq k_1}\int\frac{\langle k\rangle^{2s}\langle k-k_1\rangle^{-2l}d\tau}{\langle\tau+\alpha k^2+\epsilon^2k^4\rangle\langle|\tau-\tau_1|-\beta|k-k_1|\langle\epsilon(k-k_1)\rangle\rangle^{2b_2}}$$
$$\lesssim\sup_{\tau_1,k_1}\frac{\langle k_1\rangle^{-2s}}{\langle\tau_1+\alpha k_1^2+\epsilon^2k_1^4\rangle}\sum_{k\neq k_1;\pm}\frac{\langle k\rangle^{2s}\langle k-k_1\rangle^{-2l}}{\langle\epsilon^2k^4+\alpha k^2+\tau_1\pm\beta(k_1-k)\langle\epsilon(k-k_1)\rangle\rangle^{2b_2-}},$$

where we set $b_1 = \frac{1}{2}$. By Lemma 7, if $|k| \geq 2|k_1|$ and $|k| \geq C(\alpha,\beta,\epsilon)$, then $\langle \tau_1 + \alpha k_1^2 + \epsilon^2 k_1^4 \rangle \gtrsim \langle k \rangle^4$, and we have

$$II \lesssim \sum_{k \neq k_1; \pm} \frac{\max(1, \langle k \rangle^{-2s})}{\langle k \rangle^{2l-2s+4} \langle \epsilon^2 k^4 + \alpha k^2 + \tau_1 \pm \beta(k_1 - k)\langle \epsilon(k-k_1) \rangle \rangle^{2b_2-}} \leq c_1,$$

and similarly, the desired uniform bound of II follows if $|k| \geq 2|k_1|$ and $|k| \lesssim 1$; by applying Lemma 7 again, we can show that II is uniformly bounded for $\frac{|k_1|}{2} \geq |k|$ by treating $|k| \leq C(\alpha,\beta,\epsilon)$ and $|k| \geq C(\alpha,\beta,\epsilon)$ separately. For $\frac{|k|}{2} \leq |k_1| \leq 2|k|$, we can argue as (4.4).

III. $\max\big(|\tau + \alpha k^2 + \epsilon^2 k^4|, |\tau_1 + \alpha k_1^2 + \epsilon^2 k_1^4|, ||\tau_2| - \beta |k_2| \langle \epsilon k_2 \rangle|\big) = ||\tau_2| - \beta|k_2|\langle \epsilon k_2 \rangle|$.

From (4.1), it follows that

$$||\tau_2| - \beta|k_2|\langle \epsilon k_2\rangle| \gtrsim |k_2| \left|(2k - k_2)(\alpha + \epsilon^2(k^2 + (k-k_2)^2)) \mp \beta\langle \epsilon k_2 \rangle\right|. \quad (4.5)$$

If suffices to show $\sup_{\tau_2, k_2} III < \infty$ where $b_2 = \frac{1}{2}$ and $III :=$

$$\frac{\langle k_2 \rangle^{-2l}}{\langle |\tau_2| - \beta|k_2|\langle \epsilon k_2 \rangle\rangle} \sum_k \int \frac{\langle k \rangle^{2s} \langle k - k_2 \rangle^{-2s} d\tau}{\langle \tau + \alpha k^2 + \epsilon^2 k^4 \rangle \langle \tau - \tau_2 + \alpha(k-k_2)^2 + \epsilon^2(k-k_2)^4 \rangle^{2b_1}}$$

$$\lesssim \sum_k \frac{\langle k \rangle^{2s}}{\langle k_2 \rangle^{2l} \langle k - k_2 \rangle^{2s} \langle |\tau_2| - \beta |k_2| \langle \epsilon k_2 \rangle \rangle \langle 4\epsilon^2 k_2 p(k) \rangle^{2b_1-}},$$

where

$$p(k) = k^3 - \frac{3k_2}{2}k^2 + \Big(\frac{\alpha + 2\epsilon^2 k_2^2}{2\epsilon^2}\Big)k + \frac{\tau_2 - \alpha k_2^2 - \epsilon^2 k_2^4}{4\epsilon^2 k_2}. \quad (4.6)$$

If $\frac{2}{3}|k_2| \leq |k| \leq 2|k_2|$, then $\frac{\langle k \rangle^{2s}}{\langle k_2 \rangle^{2l} \langle k - k_2 \rangle^{2s}} \lesssim \frac{1}{\langle k \rangle^{2l-2s} \langle k - k_2 \rangle^{2s}}$. If $|k| \lesssim 1$, $III \lesssim \sigma_2 \leq c_2$ by Lemma 5. If $|k| \gg 1$, we argue as in Lemma 7 to obtain

$$\left|(2k - k_2)(\alpha + \epsilon^2(k^2 + (k-k_2)^2)) \mp \beta \langle \epsilon k_2 \rangle\right| \gtrsim |k|^3,$$

from which, we estimate

$$III \lesssim \sum_k \frac{\max(1, \langle k \rangle^{-2s})}{\langle k \rangle^{2l-2s+4} \langle k_2 p(k) \rangle^{2b_1-}} \lesssim \sigma_2 \leq c_2,$$

by Lemma 5 and $l \geq -2$.

If $|k| \leq \frac{2}{3}|k_2|$ or $|k| \geq 2|k_2|$, then $\frac{\langle k \rangle^{2s}}{\langle k_2 \rangle^{2l} \langle k - k_2 \rangle^{2s}} \lesssim \frac{1}{\langle k_2 \rangle^{2l}}$ since $|k - k_2| \geq \frac{|k|}{2}$. As in Lemma 6, if $|k_2| \geq C(\alpha,\beta,\epsilon)$, we have

$$III \lesssim \sum_k \frac{1}{\langle k_2 \rangle^{2l+2} \langle k_2 p(k) \rangle^{2b_1-}} \lesssim \sigma_2 \leq c_2.$$

Lastly if $|k_2| \lesssim 1$ (for $|k| \leq \frac{2}{3}|k_2|$) or $|k| \lesssim 1$ (for $|k| \geq 2|k_2|$), then $\langle k_2 \rangle \simeq 1$ and we have $III \lesssim \sigma_2 \leq c_2$, which concludes the proof of the first inequality of Proposition 1. To show the second inequality by the duality argument, it suffices to estimate

$$\sum_{k_1+k_2-k=0}\int_{\tau_1+\tau_2-\tau=0}\frac{\langle k\rangle^{l+\rho}\langle k_1\rangle^{-s}\langle k_2\rangle^{-s}f(\tau_1,k_1)g(-\tau_2,-k_2)w(\tau,k)}{\langle|\tau|-\beta|k|\langle\epsilon k\rangle\rangle^{1/2}\langle\tau_1+\alpha k_1^2+\epsilon^2k_1^4\rangle^{b_1}\langle\tau_2-\alpha k_2^2-\epsilon^2k_2^4\rangle^{b_2}},$$

where $f(\tau,k)=|\hat{u}(\tau,k)|\langle k\rangle^s\langle\tau+\alpha k^2+\epsilon^2k^4\rangle^{b_1}$, $g(\tau,k)=|\hat{v}(\tau,k)|\langle k\rangle^s\langle\tau+\alpha k^2+\epsilon^2k^4\rangle^{b_2}$ and $b_1,b_2\le\frac{1}{2}$. Since $\rho>0$, we take $k\neq 0$ in the sum.

IV. $\max\big(\,||\tau|-\beta|k|\langle\epsilon k\rangle|\,,|\tau_1+\alpha k_1^2+\epsilon^2k_1^4|,|\tau_2-\alpha k_2^2-\epsilon^2k_2^4|\big)=||\tau|-\beta|k|\langle\epsilon k\rangle|$.

In this region, the lower bound of the dispersive weight is similar to (4.5). For $b_1=b_2=b=\frac{1}{2}-$, it suffices to show

$$\begin{aligned}\sup_{\tau,k}IV&:=\sup_{\tau,k}\frac{\langle k\rangle^{2l+2\rho}}{\langle|\tau|-\beta|k|\langle\epsilon k\rangle\rangle}\\&\quad\times\sum_{k_1}\int\frac{\langle k_1\rangle^{-2s}\langle k-k_1\rangle^{-2s}d\tau_1}{\langle\tau_1+\alpha k_1^2+\epsilon^2k_1^4\rangle^{2b}\langle\tau-\tau_1-\alpha(k-k_1)^2-\epsilon^2(k-k_1)^4\rangle^{2b}}\\&\lesssim\sup_{\tau,k}\frac{\langle k\rangle^{2l+2\rho}}{\langle|\tau|-\beta|k|\langle\epsilon k\rangle\rangle}\sum_{k_1}\frac{1}{\langle k_1\rangle^{2s}\langle k-k_1\rangle^{2s}\langle\langle k\rangle p(k_1)\rangle^{4b-1}}<\infty,\end{aligned}\tag{4.7}$$

where p is defined in (4.6). If $\frac{2}{5}|k|\le|k_1|\le\frac{2}{3}|k|$, then $\frac{|k|}{3}\le|k-k_1|\le\frac{5}{3}|k|$ and $\frac{\langle k\rangle^{2l+2\rho}}{\langle k_1\rangle^{2s}\langle k-k_1\rangle^{2s}}\lesssim\frac{1}{\langle k\rangle^{4s-2l-2\rho}}$. For $|k|\lesssim 1$, (4.7) reduces to Lemma 5. If $|k|\ge C(\alpha,\beta,\epsilon)$ as in Lemma 6,

$$IV\lesssim\sum_{k_1}\frac{1}{\langle k\rangle^{4s-2l-2\rho+2}\langle\langle k\rangle p(k_1)\rangle^{4b-1}}\lesssim\sigma_2,$$

since $4s-2l-2\rho+2\ge 0$. If $\frac{2}{3}|k|\le|k_1|\le\frac{3}{2}|k|$ and $|k|\lesssim 1$, then again (4.7) reduces to Lemma 5. If $|k|\ge C(\alpha,\beta,\epsilon)$, then as in Lemma 7,

$$IV\lesssim\sum_{k_1}\frac{1}{\langle k\rangle^{2s-2l-2\rho+4}\langle\langle k\rangle p(k_1)\rangle^{4b-1}}\le\sigma_2,\tag{4.8}$$

since $s-l\ge-2+\rho$. Similarly for $|k_1|\le\frac{2}{5}|k|$ or $|k_1|\ge\frac{3}{2}|k|$, we treat $|k|\lesssim 1$ and $|k|\ge C(\alpha,\beta,\epsilon)$ separately where for $|k|\ge C(\alpha,\beta,\epsilon)$, we have $\langle|\tau-\beta|k|\langle\epsilon k\rangle|\rangle\gtrsim\langle k\rangle^4$, and therefore we can argue as (4.8).

V. $\max\big(\,||\tau|-\beta|k|\langle\epsilon k\rangle|\,,|\tau_1+\alpha k_1^2+\epsilon^2k_1^4|,|\tau_2-\alpha k_2^2-\epsilon^2k_2^4|\big)=|\tau_1+\alpha k_1^2+\epsilon^2k_1^4|$.

It suffices to show

$$\begin{aligned}\sup_{\tau_1,k_1}V&:=\sup_{\tau_1,k_1}\frac{\langle k_1\rangle^{-2s}}{\langle\tau_1+\alpha k_1^2+\epsilon^2k_1^4\rangle}\\&\quad\times\sum_{k}\int\frac{\langle k\rangle^{2l+2\rho}\langle k-k_1\rangle^{-2s}d\tau}{\langle|\tau|-\beta|k|\langle\epsilon k\rangle\rangle\langle\tau-\tau_1-\alpha(k-k_1)^2-\epsilon^2(k-k_1)^4\rangle^{2b_2}}\\&\lesssim\sup_{\tau_1,k_1}\frac{\langle k_1\rangle^{-2s}}{\langle\tau_1+\alpha k_1^2+\epsilon^2k_1^4\rangle}\sum_{k,\pm}\frac{\langle k\rangle^{2l+2\rho}\langle k-k_1\rangle^{-2s}}{\langle\epsilon^2(k-k_1)^4+\alpha(k-k_1)^2+\tau_1\mp\beta k\langle\epsilon k\rangle\rangle^{2b_2-}},\end{aligned}$$

where $|\tau_1+\alpha k_1^2+\epsilon^2 k_1^4| \gtrsim |k|\cdot|\alpha(2k_1-k)(\alpha+\epsilon^2(k_1^2+(k-k_1)^2))\mp\beta\langle\epsilon k\rangle|$ and $b_1=\frac{1}{2}$. Note that for
$\max\left(||\tau|-\beta|k|\langle\epsilon k\rangle|, |\tau_1+\alpha k_1^2+\epsilon^2k_1^4|, |\tau_2-\alpha k_2^2-\epsilon^2k_2^4|\right)=|\tau_2-\alpha k_2^2-\epsilon^2 k_2^4|$, the corresponding $L^\infty L^1$ estimate reduces to the current case by an appropriate change of variable.

If $|k|\le\frac{|k_1|}{2}$ or $\frac{3|k_1|}{2}\le|k|\le\frac{5|k_1|}{2}$, then $|k-k_1|\gtrsim|k_1|$, and therefore $\langle k_1\rangle^{-2s}\langle k\rangle^{2l+2\rho}\langle k-k_1\rangle^{-2s}\lesssim\langle k_1\rangle^{-4s}\langle k\rangle^{2l+2\rho}$. Hence $V\lesssim\sigma_1$ if $|k_1|\lesssim 1$, and by Lemma 6,

$$V\lesssim\sum_{k,\pm}\frac{\langle k_1\rangle^{-4s}\max(1,\langle k_1\rangle^{2l+2\rho-2})}{\langle\epsilon^2(k-k_1)^4+\alpha(k-k_1)^2+\tau_1\mp\beta k\langle\epsilon k\rangle\rangle^{2b_2-}}\lesssim\sigma_1,$$

if $|k_1|\ge C(\alpha,\beta,\epsilon)$. If $\frac{|k_1|}{2}\le|k|\le\frac{3|k_1|}{2}$, then $\langle k_1\rangle^{-2s}\langle k\rangle^{2l+2\rho}\langle k-k_1\rangle^{-2s}\lesssim\frac{1}{\langle k\rangle^{2s-2l-2\rho}}$. If $|k|\lesssim 1$, then $V\lesssim\sigma_1$, and by Lemma 7 if $|k|\ge C(\alpha,\beta,\epsilon)$, then $V\lesssim\sigma_1$ since $2s-2l-2\rho+4\ge 0$. A similar statement follows for $\frac{5|k_1|}{2}\le|k|$ if $s-l\ge-2+\rho$ since $|k-k_1|\gtrsim|k|$ and $\langle\tau_1+\alpha k_1^2+\epsilon^2k_1^4\rangle\gtrsim\langle k\rangle^4$ for sufficiently large $|k|$ by Lemma 7. □

In the Appendix, we give explicit examples to show the converse statement for Proposition 1 (see also [15]). Hence, Proposition 1 is sharp up to the boundary based on our method as long as $s\ge 0$.

Proposition 3. *Suppose* $\|un\|_{X_S^{s,b-1}}\lesssim\|u\|_{X_S^{s,b}}\|n\|_{X_W^{l,b}}$ *holds for all* $u,n\in C_c^\infty(\mathbb{T}\times\mathbb{R})$ *for some* $s,l,b\in\mathbb{R}$. *Then* $l\ge\max(2(b-1),-2b)\ge-1$ *and* $s-l\le\min(-4(b-1),4b)\le 2$. *Furthermore, suppose* $\|D^\rho(u\overline{v})\|_{X_W^{l,b-1}}\lesssim\|u\|_{X_S^{s,b}}\|v\|_{X_S^{s,b}}$ *holds for all* $u,v\in C_c^\infty(\mathbb{T}\times\mathbb{R})$ *for some* $s,l,b\in\mathbb{R}$, $\rho\in(0,1]$. *Then* $2s-l-\rho\ge\max(2(b-1),-2b)\ge-1$ *and* $s-l\ge\max(\rho+4(b-1),\rho-4b)\ge-2+\rho$.

5 Global well-posedness and semi-classical limit

We adopt the argument of [8] to show that a (strict) subset of local solutions obtained in Theorem 1 can be extended globally; however it is suggested from the scaling-invariance perspectives in [7] that any local solution is global. Let $\alpha=\beta=1$ for the sake of simplicity. To prove Theorem 2, the following Sobolev inequality is useful.

Lemma 8. *Let* $d\in\mathbb{N}$, $s\in[-\frac{d}{2},\frac{d}{2}]$ *and consider* $H^s(M)$, *where* $M=\mathbb{R}^d,\mathbb{T}^d$. *Then,*

$$\|fg\|_{H^s}\lesssim_{d,s}\|f\|_{H^{\frac{d}{2}+}}\|g\|_{H^s}.$$

Proof. If $s=0$, the statement follows from the Hölder's inequality and the Sobolev embedding $H^{\frac{d}{2}+}\hookrightarrow L^\infty$. If $s<0$, then

$$\|fg\|_{H^s}=\sup_{\|h\|_{H^{-s}}=1}|\langle fg,h\rangle|\le\|g\|_{H^s}\sup_{\|h\|_{H^{-s}}=1}\|fh\|_{H^{-s}},$$

and hence it suffices to show the statement for $s > 0$. By the Leibniz's rule,

$$\|fg\|_{H^s} \lesssim \|f\|_{W^{s,q}} \|g\|_{L^r} + \|f\|_{L^\infty} \|g\|_{H^s},$$

where $q \in [2, \infty), r \in (2, \infty]$ are to be determined. The second term is bounded above by $\|f\|_{H^{\frac{d}{2}+}} \|g\|_{H^s}$ again by the Sobolev embedding. To obtain $H^{\frac{d}{2}+} \hookrightarrow W^{s,q}$, $H^s \hookrightarrow L^r$, it suffices to have $\frac{1}{2} - \frac{1}{q} < \frac{(d/2+)-s}{d}$, $\frac{1}{2} - \frac{1}{r} < \frac{s}{d}$. By noting $\frac{1}{2} = \frac{1}{q} + \frac{1}{r}$, we can pick $r \in (2, \infty]$ such that $\frac{1}{2} - \frac{s}{d} < \frac{1}{r} < \frac{(d/2+)-s}{d}$, which uniquely determines $q \in [2, \infty)$, and therefore validates the desired Sobolev embedding. □

The proof of Theorem 2 makes use of Lemma 8 and the Gronwall's inequality.

Proof. Assume $(s, l) = (2, 1)$. By the Gagliardo-Nirenberg inequality, (1.2), and the Young's inequality,

$$\left| \int n|u|^2 \right| \leq \frac{1}{4} \|n\|_{L^2}^2 + \frac{\epsilon^2}{2} \|\partial_x u\|_{L^2}^2 + C(\|u_0\|_{L^2}, \epsilon), \tag{5.1}$$

and since

$$\begin{aligned} &\|u(t)\|_{H^2}^2 + \|n(t)\|_{H^1}^2 + \|\partial_t n(t)\|_{H^{-1}}^2 \lesssim \|u_0\|_{L^2}^2 + \|\partial_{xx} u(t)\|_{L^2}^2 + \|n(t)\|_{L^2}^2 \\ &+ \|\partial_x n\|_{L^2}^2 + \|\partial_t n(t)\|_{\dot{H}^{-1}}^2 \lesssim \|u_0\|_{L^2}^2 + (1+\epsilon^{-2})|H_0| + (1+\epsilon^{-2}) \left| \int n|u|^2 \right|, \end{aligned} \tag{5.2}$$

we obtain $\|(u, n, \partial_t n)\|_{H^{2,1}} \leq C(\epsilon)$ for all $t \in \mathbb{R}$ by using (5.1) to absorb $\|n\|_{L^2}, \|\partial_x u\|_{L^2}$ to the LHS of (5.2).

Let $(s, l) \in \Omega_G \setminus \{(2, 1)\}$ and denote $l = s - l_0$ for $0 \leq l_0 \leq 2$, where since $s + l \geq 4$, it follows that

$$s \geq 2 + \frac{l_0}{2} \geq 2, \; l \geq 2 - \frac{l_0}{2} \geq 1.$$

With $a = l - 2$, multiply $\langle \nabla \rangle^{2a} \partial_t n$ to the second equation of (1.1) and integrate by parts to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|\partial_t n\|_{H^a}^2 + \|\partial_x n\|_{H^a}^2 + \epsilon^2 \|\partial_{xx} n\|_{H^a}^2 \right) &= \int (\langle \nabla \rangle^a \partial_t n)(\langle \nabla \rangle^a \partial_{xx} |u|^2) \\ &\lesssim_a \|\partial_t n\|_{H^a}^2 + \|u\|_{H^{a+2}}^2. \end{aligned} \tag{5.3}$$

First, let $l_0 > 0$. For $T > 0$, assume the inductive hypothesis, $\|u\|_{H^l} \leq C(T, l_0, \epsilon) < \infty$, from which the Gronwall's inequality on (5.3) yields

$$\|\partial_t n\|_{H^a}^2 + \|\partial_x n\|_{H^a}^2 + \epsilon^2 \|\partial_{xx} n\|_{H^a}^2 \leq C(T),$$

which, together with (1.2), controls $\|n\|_{H^l}$.

Now take ∂_t of the first equation of (1.1), multiply the resulting equation by $-i\langle \nabla \rangle^{2b} \overline{\partial_t u}$, integrate by parts, and take its real part to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_t u\|_{H^b}^2 &= Im \int \left(\langle \nabla \rangle^b \overline{\partial_t u} \right) \left(\langle \nabla \rangle^b (\partial_t u \cdot n + u \cdot \partial_t n) \right) \\ &\leq \|\partial_t u\|_{H^b} (\|\partial_t u \cdot n\|_{H^b} + \|u \cdot \partial_t n\|_{H^b}). \end{aligned} \tag{5.4}$$

The first equation of (1.1) is equivalent to

$$\Delta u = \langle \epsilon \nabla \rangle^{-2}(-i\partial_t u + un).$$

Let $b = s - 4$. Note that $\|\Delta u\|_{H^{s-2}}$ controls $\|u\|_{H^s}$ by (1.2). We claim

$$\|\partial_t u \cdot n\|_{H^b} \lesssim \|\partial_t u\|_{H^b} \|n\|_{H^l}.$$

If $s > \frac{9}{2}$, then $b > \frac{1}{2}$ and $\|\partial_t u \cdot n\|_{H^b} \lesssim \|\partial_t u\|_{H^b}\|n\|_{H^b} \leq \|\partial_t u\|_{H^b}\|n\|_{H^{s-4}}$. On the other hand, if $s < \frac{7}{2}$, then $-b > \frac{1}{2}$ and

$$\begin{aligned}\|\partial_t u \cdot n\|_{H^b} = & \sup_{\|\phi\|_{H^{-b}}=1} |\langle \partial_t u \cdot n, \phi \rangle| \leq \|\partial_t u\|_{H^b} \sup_{\|\phi\|_{H^{-b}}=1} \|n\phi\|_{H^{-b}} \\ & \lesssim \|\partial_t u\|_{H^b}\|n\|_{H^{-b}} \leq \|\partial_t u\|_{H^b}\|n\|_{H^l},\end{aligned}$$

where the last inequality holds since $s \geq 2 + \frac{l_0}{2}$. If $\frac{7}{2} \leq s \leq \frac{9}{2}$, then $-\frac{1}{2} \leq b \leq \frac{1}{2}$ and by Lemma 8,

$$\|\partial_t u \cdot n\|_{H^b} \lesssim \|\partial_t u\|_{H^b}\|n\|_{H^l}.$$

Similarly $\|u \cdot \partial_t n\|_{H^b} \lesssim 1$, and hence $\frac{d}{dt}\|\partial_t u\|^2_{H^b} \lesssim (\|\partial_t u\|^2_{H^b} + \|\partial_t u\|_{H^b})$, from which the Gronwall's inequality yields $\|\partial_t u\|_{H^{s-4}} \leq C(T)$. Using similar arguments, we obtain $\|un\|_{H^b} \leq C(T)$, and from

$$\|\Delta u\|_{H^{s-2}} \leq \|\langle \epsilon \nabla \rangle^{-2} \partial_t u\|_{H^{s-2}} + \|\langle \epsilon \nabla \rangle^{-2}(un)\|_{H^{s-2}}$$

follows $\|u\|_{H^s} \leq C(T)$. To show the inductive hypothesis, consider the base case $s_0 = 2 + \frac{l_0}{2}$. Then $\|u\|_{H^{2-\frac{l_0}{2}}} \leq C$ by (1.2). Then for all $s \in [s_0, s_1]$, where $s_1 = s_0 + l_0$, it follows that $\|u\|_{H^{s-l_0}} \leq \|u\|_{H^{s_0}}$. This process is iterated by an increment of l_0 to cover the entire range of $s \geq 2 + \frac{l_0}{2}$. It remains to prove the $l_0 = 0$ case.

Let $s \geq 2 + \frac{\epsilon_0}{2}$, where $0 \leq \epsilon_0 \leq 1$. As before, consider the energy estimate

$$\begin{aligned}&\frac{d}{dt}\Big(\|\partial_t n\|^2_{H^{s-2}} + \|\partial_x n\|^2_{H^{s-2}} + \epsilon^2\|\partial_{xx} n\|^2_{H^{s-2}}\Big) \lesssim \|\partial_t n\|^2_{H^{s-2}} + \|u\|^2_{H^s} \\ &\frac{d}{dt}\|\partial_t u\|^2_{H^{s-4}} \lesssim \|\partial_t u\|_{H^{s-4}}(\|\partial_t u \cdot n\|_{H^{s-4}} + \|u \cdot \partial_t n\|_{H^{s-4}}),\end{aligned} \tag{5.5}$$

where by a similar argument as before

$$\|\partial_t u \cdot n\|_{H^{s-4}} \lesssim \|\partial_t u\|_{H^{s-4}}\|n\|_{H^{s-\epsilon_0}};\ \|u\partial_t n\|_{H^{s-4}} \lesssim \|u\|_{H^s}\|\partial_t n\|_{H^{s-4}}.$$

Recall

$$\|\Delta u\|_{H^{s-2}} \lesssim \|\partial_t u\|_{H^{s-4}} + \|un\|_{H^{s-4}} \lesssim \|\partial_t u\|_{H^{s-4}} + \|u\|_{H^{s-\epsilon_0}}\|n\|_{H^{s-\epsilon_0}},$$

and hence

$$\begin{aligned}&\frac{d}{dt}\Big(\|\partial_t n\|^2_{H^{s-2}} + \|\partial_x n\|^2_{H^{s-2}} + \epsilon^2\|\partial_{xx} n\|^2_{H^{s-2}}\Big) \\ &\quad \lesssim \|\partial_t n\|^2_{H^{s-2}} + \|\partial_t u\|^2_{H^{s-4}} + \|u_0\|^2_{L^2} + (\|u\|_{H^{s-\epsilon_0}}\|n\|_{H^{s-\epsilon_0}})^2, \\ &\frac{d}{dt}\|\partial_t u\|^2_{H^{s-4}} \lesssim \|\partial_t u\|^2_{H^{s-4}}\|n\|_{H^{s-\epsilon_0}} \\ &\quad + \|\partial_t u\|_{H^{s-4}}(\|u_0\|_{L^2} + \|\partial_t u\|_{H^{s-4}} + \|u\|_{H^{s-\epsilon_0}}\|n\|_{H^{s-\epsilon_0}})\|\partial_t n\|_{H^{s-4}}.\end{aligned}$$

If $\epsilon_0 = 0$, then integrate the first differential inequality of (5.5) to obtain an exponential growth bound on $\|n\|_{H^2} + \|\partial_t n\|_{L^2}$, and then apply the Gronwall's inequality again to the second differential inequality of (5.5). If $s > 2$, use the exponential growth bound for $s = 2$ for the base case $s_0 = 2 + \frac{\epsilon_0}{2}$. Such exponential bound is obtained for all $s \geq 2 + \frac{\epsilon_0}{2}$ by iterating the Gronwall's inequality. Since $\epsilon_0 > 0$ is arbitrary, we have an exponential bound on the Sobolev norms of solutions for all $s \geq 2$. □

To discuss the semi-classical limit to ZS, denote $(u^\epsilon, n^\epsilon, \partial_t n^\epsilon)$ by the QZS solution with data $(u_0^\epsilon, n_0^\epsilon, n_1^\epsilon)$ for $\epsilon \geq 0$. Given a solution $(u^\epsilon, n^\epsilon, \partial_t n^\epsilon)$, we denote H^ϵ by the corresponding energy and H_0^ϵ by H^ϵ at $t = 0$. Define $H_0^{s,l} = H^s(\mathbb{T}) \times H^l(\mathbb{T}) \times H^{l-1}(\mathbb{T})$.

Proposition 4. *Let $s \geq 4$. If $\sup_\epsilon \|(u_0^\epsilon, n_0^\epsilon, n_1^\epsilon)\|_{H^{s,s-1}} \leq R < \infty$ and $(u_0^0, n_0^0, n_1^0) \in H_0^{s,s-1}$ where $(u_0^\epsilon, n_0^\epsilon, n_1^\epsilon) \xrightarrow[\epsilon \to 0]{H_0^{s-2,s-3}} (u_0^0, n_0^0, n_1^0)$, then $(u^\epsilon, n^\epsilon, \partial_t n^\epsilon) \xrightarrow[\epsilon \to 0]{} (u^0, n^0, \partial_t n^0)$ in $C([0,T]; H_0^{s-2,s-3})$.*

To show Proposition 4, two key hypotheses need to be verified, after which the proof proceeds as [8, theorem 1.3] and thus is omitted. First, a uniform bound on $(u^\epsilon, n^\epsilon, \partial_t n^\epsilon)$ that depends only on R, T is needed. Second, the space of data needs to be regularized. In [8], this is done via a particular convolution kernel on $\mathbb{R}^d$. On a periodic domain, we define a family of mollifiers as follows: for $h > 0$, define $\widehat{J_h f}(k) = \eta(hk)\widehat{f}(k)$ for all $f \in L^1(\mathbb{T})$. Then $\|J_h f - f\|_{H^s} \xrightarrow[h \to 0]{} 0$ and for $\sigma > 0$

$$\|J_h f - f\|_{H^{s-\sigma}} \leq C(\sigma) h^\sigma \|f\|_{H^s}, \; \|J_h f\|_{H^{s+\sigma}} \leq \frac{C(\sigma)}{h^\sigma}\|f\|_{H^s}.$$

It suffices to obtain a uniform bound on $(u^\epsilon, n^\epsilon, \partial_t n^\epsilon)$.

Lemma 9. *If $(s,l) \in \Omega_G$ and $\sup_{\epsilon > 0} \|(u_0^\epsilon, n_0^\epsilon, n_1^\epsilon)\|_{H^{s,l}} \leq R < \infty$, then $\sup_{\epsilon>0} \sup_{t \in [0,\infty)} \|(u^\epsilon, n^\epsilon, \partial_t n^\epsilon)\|_{H_0^{1,0}} \leq C(R)$. If $s \geq 4$, then for all $1 \leq s' \leq s-2$ and $T > 0$, $\sup_{\epsilon > 0} \|(u^\epsilon, n^\epsilon, \partial_t n^\epsilon)\|_{C_T H_0^{s',s'-1}} \leq C(T,R)$.*

Proof. By inspection, $|H_0^\epsilon| \leq C(R)$ uniformly in ϵ. Since mass is conserved and

$$\begin{aligned} \|\partial_x u^\epsilon\|_{L^2}^2 + \frac{1}{2}\|n^\epsilon\|_{L^2}^2 + \frac{1}{2}\|\partial_t n^\epsilon\|_{H^{-1}} &\leq |H_0^\epsilon| + \left| \int n^\epsilon |u^\epsilon|^2 \right| \\ &\leq |H_0^\epsilon| + \frac{1}{4}\|n^\epsilon\|_{L^2}^2 + \frac{1}{2}\|\partial_x u^\epsilon\|_{L^2}^2 + C', \end{aligned}$$

where the last inequality is by the Gagliardo-Nirenberg inequality, and C' is independent of ϵ, we obtain the first uniform bound. Now assume $s \geq 4, T > 0$ and the following inductive hypotheses:

$$\|u^\epsilon\|_{H^{s'-2}}, \|n^\epsilon\|_{H^{s'-2}}, \|n^\epsilon\|_{H^1} \leq C(T,R), \tag{5.6}$$

uniformly in $\epsilon > 0$ and $t \in [0,T]$. Then (1.2) and (5.6) yield $\|u^\epsilon n^\epsilon\|_{H^{s'-2}} \lesssim C(T,R)$ since

$$\|u^\epsilon n^\epsilon\|_{H^{s'-2}} \lesssim \|u^\epsilon\|_{H^{s'-2}}\|n^\epsilon\|_{H^{s'-2}} \le C(T,R)$$

for $s' > \frac{5}{2}$ and

$$\|u^\epsilon n^\epsilon\|_{H^{s'-2}} \le \|u^\epsilon n^\epsilon\|_{H^1} \lesssim \|u^\epsilon\|_{H^1}\|n^\epsilon\|_{H^1} \le C(T,R)$$

for $s' \in [1, \frac{5}{2}]$. Moreover, since $\|u^\epsilon\|_{\dot H^{s'}} \le \|\langle\epsilon\nabla\rangle^2 \Delta u^\epsilon\|_{H^{s'-2}}$ for all $\epsilon \ge 0$,

$$\|u^\epsilon\|^2_{H^{s'}} \lesssim \|u^\epsilon_0\|^2_{L^2} + \|\partial_t u^\epsilon\|^2_{H^{s'-2}} + \|u^\epsilon n^\epsilon\|^2_{H^{s'-2}} \le \|u^\epsilon_0\|^2_{L^2} + \|\partial_t u^\epsilon\|^2_{H^{s'-2}} + C(T,R),$$

and hence the differential inequality obtained from the first equation of (1.1) is

$$\begin{aligned} \frac{d}{dt}\Big(\|\partial_t n^\epsilon\|^2_{H^{s'-2}} + \|\partial_x n^\epsilon\|^2_{H^{s'-2}} + \epsilon^2\|\partial_{xx} n^\epsilon\|^2_{H^{s'-2}}\Big) \\ \lesssim \|\partial_t n^\epsilon\|^2_{H^{s'-2}} + \|\partial_t u^\epsilon\|^2_{H^{s'-2}} + C(T,R), \end{aligned} \tag{5.7}$$

where its LHS is well-defined since $s' \le s-2 \le l$ from $(s,l) \in \Omega_G$.

Similarly we derive another differential inequality as (5.4) with $b = s'-2$. A similar calculation as before shows

$$\|\partial_t u^\epsilon \cdot n^\epsilon\|_{H^{s'-2}} \lesssim \|\partial_t u^\epsilon\|_{H^{s'-2}},\ \|u^\epsilon \partial_t n^\epsilon\|_{H^{s'-2}} \lesssim \|\partial_t n^\epsilon\|_{H^{s'-2}},$$

by the inductive hypothesis where the implicit constants are independent of ϵ, and hence by the Young's inequality

$$\frac{d}{dt}\|\partial_t u^\epsilon\|^2_{H^{s'-2}} \lesssim \|\partial_t u^\epsilon\|^2_{H^{s'-2}} + \|\partial_t n^\epsilon\|^2_{H^{s'-2}}. \tag{5.8}$$

Integrating (5.7) and (5.8),

$$\|\partial_t n^\epsilon\|^2_{H^{s'-2}} + \|\partial_x n^\epsilon\|^2_{H^{s'-2}} + \epsilon^2\|\partial_{xx} n^\epsilon\|^2_{H^{s'-2}} + \|\partial_t u^\epsilon\|^2_{H^{s'-2}} \le C(T,R).$$

Now we show (5.6). By [8, proposition 2.4], we have $\sup_\epsilon \|n^\epsilon\|_{C_T H^1_x} \le C(T,R)$. On the other hand, if $1 \le s' \le 2$, then $\|u^\epsilon\|_{H^{s'-2}}, \|n^\epsilon\|_{H^{s'-2}} \le C$, independent of ϵ, by Lemma 9. Hence for such s', Lemma 9 holds and using $s'_0 = 2$ as a base case, we can extend the uniform bound to all $2 \le s' \le 3$ from which we iterate to cover the entire $1 \le s' \le s-2$. □

Remark 3. When $T = \infty$, the continuity of $\epsilon \to u^\epsilon$ is not expected for all $\epsilon \ge 0$. Let $(u_0, n_0, n_1) = (\langle N\rangle^{-s} e^{iNx}, 0, 0)$ for $N \in \mathbb{R} \setminus \{0\}$. Then, $(u, n, \partial_t n)(x,t) = (\langle N\rangle^{-s} e^{-it(N^2+\epsilon^2 N^4)+iNx}, 0, 0)$ is the (classical) solution. By direct computation,

$$\begin{aligned} &\sup_{t\in[0,\infty)} \|\langle N\rangle^{-s} e^{-it(N^2+\epsilon^2N^4)+iNx} - \langle N\rangle^{-s} e^{-it(N^2+\epsilon_0^2N^4)+iNx}\|_{H^s_x} \\ &\quad = \sup_{t\in[0,\infty)} |1 - e^{it(\epsilon^2-\epsilon_0^2)N^4}| = 2. \end{aligned}$$

6 Conclusions

The QZS with the periodic boundary condition is well-posed for data with low Sobolev regularity. The smoothing effect of the fourth-order perturbation nullifies the resonance phenomenon ($\frac{\beta}{\alpha} \in \mathbb{Z}$), which played an important role when $\epsilon = 0$. The continuity of solution in ϵ holds for $T < \infty$ but not for $T = \infty$.

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Appendix

We show Proposition 3 by providing concrete examples.

Proof. The main idea is to add a fourth-order perturbation to the spacetime functions constructed in [15]. Once those examples are given, one can directly substitute the examples to the inequalities in Proposition 3 to derive a set of necessary conditions on the scaling parameter $N \gg 1$. Let $\delta(k)$ be the Kronecker delta function defined on $\mathbb{Z}$ and let $\phi(\tau)$ be a smooth bump function on $\mathbb{R}$ with a compact support. It suffices to consider u_i, $1 \leq i \leq 8$, n_i, $1 \leq i \leq 4$, and v_i, $5 \leq i \leq 8$, where

$$
\begin{aligned}
&\widehat{u}_1(k,\tau) = \delta(k+N)\phi(\tau + \alpha N^2 + \epsilon^2 N^4),\\
&\widehat{n}_1(k,\tau) = \delta(k-2N)\phi(|\tau| - 2\beta N\langle 2\epsilon N\rangle),\\
&\widehat{u}_2(k,\tau) = \delta(k+N)\phi(\tau + \alpha N^2 + \epsilon^2 N^4 + 2\beta N\langle 2\epsilon N\rangle),\\
&\widehat{n}_2(k,\tau) = \delta(k-2N)\phi(|\tau| - 2\beta N\langle 2\epsilon N\rangle),\\
&\widehat{u}_3(k,\tau) = \delta(k)\phi(\tau), \quad \widehat{n}_3(k,\tau) = \delta(k-N)\phi(|\tau| - \beta N\langle \epsilon N\rangle),\\
&\widehat{u}_4(k,\tau) = \delta(k)\phi(\tau + \alpha N^2 + \epsilon^2 N^4 + \beta N\langle \epsilon N\rangle),\\
&\widehat{n}_4(k,\tau) = \delta(k-N)\phi(|\tau| - \beta N\langle \epsilon N\rangle),\\
&\widehat{u}_5(k,\tau) = \delta(k-N)\phi(\tau + \alpha N^2 + \epsilon^2 N^4),\\
&\widehat{v}_5(k,\tau) = \delta(k+N)\phi(\tau + \alpha N^2 + \epsilon^2 N^4),\\
&\widehat{u}_6(k,\tau) = \delta(k-N)\phi(\tau + \alpha N^2 + \epsilon^2 N^4 - 2\beta N\langle 2\epsilon N\rangle),\\
&\widehat{v}_6(k,\tau) = \delta(k+N)\phi(\tau + \alpha N^2 + \epsilon^2 N^4),\\
&\widehat{u}_7(k,\tau) = \delta(k)\phi(\tau), \quad \widehat{v}_7(k,\tau) = \delta(k-N)\phi(\tau + \alpha N^2 + \epsilon^2 N^4),\\
&\widehat{u}_8(k,\tau) = \delta(k)\phi(\tau + \alpha N^2 + \epsilon^2 N^4 + \beta N\langle \epsilon N\rangle),\\
&\widehat{v}_8(k,\tau) = \delta(k-N)\phi(\tau + \alpha N^2 + \epsilon^2 N^4).
\end{aligned}
$$

□