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# Virtual Element Approximations for Two Species Model of the Advection-Diffusion-Reaction in Poroelastic Media

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**Abstract.** This paper proposes virtual element methods for approximating the mathematical model consisting of coupled poroelastic and Advection-Diffusion-Reaction (ADR) equations. The space discretization relies on virtual element spaces containing piecewise linear polynomials as well as non-polynomials for displacement, pressure and concentrations, and piecewise constant for total pressure; a backward-Euler scheme is employed for the approximation of time derivative. Using standard techniques of explicit schemes, the well-posedness of the resultant fully discrete scheme is derived. Moreover, under certain regularity assumptions on the mesh, optimal *a-priori* error estimates are established by introducing suitable projection operators. Several numerical experiments are presented to validate the theoretical convergence rate and exhibit the proposed scheme's performance.

**Keywords:** poroelastic equation, advection-diffusion-reaction equation, virtual element method, lowest order, inf-sup condition, numerical experiments, convergence analysis.

**AMS Subject Classification:** 65B99; 76S99; 76R99; 76M30; 65M99.

## 1 Introduction

Considering the computational efficiency of polygonal meshes, in the last decade, there have been many developments in virtual element methods (VEMs)– numerical techniques evolved from mimetic finite difference methods, see [1,5] and references therein. Contrary to finite element methods, in VEM, the mass

and stiffness matrices are computed (with the help of degrees of freedom) without explicit knowledge of the basis functions. This is desirable while dealing with polygonal meshes and demanding more accurate solutions, i.e., higher-order approximations. Virtual element spaces involved in the discrete formulations contain the polynomial functions and non-polynomial functions (solution of suitable PDEs). However, neither the implementation nor the theory demands the expression of these unknown non-polynomial functions. The presence of the polynomial functions in the virtual element spaces would help in demonstrating the convergence rates of the proposed VEM. We remark that convergence analysis of VEM can be carried out similarly as in finite element methods by introducing certain projection operators onto the space of polynomial functions. Recently, VEM has been also developed in [12, 14, 24, 25, 26] for the Biot's equation and for nonlinear problems in [9, 13].

In this paper, we propose a virtual element method for a three-field formulation of the time-dependent poromechanics equations coupled with advection-diffusion-reaction equations for two species. We base the development of current work as the formulation proposed in [20] and [22] for the stationary Biot system and extend the discrete analysis to include the quasi-steady case, and the development in time-dependent problems [9, 27]. Several other works on the Biot's equation with various numerical techniques such as finite element method, hybrid high order method, discontinuous Galerkin method are seen in [10, 11, 19, 23, 30] and references within.

## Governing equations: Coupled poroelastic and a system of advection-diffusion-reaction equations

Let us consider the model describing the fluid flow of two chemical species in poroelastic media on a domain $\Omega \subset \mathbb{R}^2$. The coupled problem is stated as follows [15, 28]: For given body load $\boldsymbol{b}$ and a mass source $\ell$, one seeks for each time $t \in (0, t_{\text{final}}]$, the displacements of the porous skeleton $\boldsymbol{u}^s$, the pore pressure of the fluid $p^f$, the volumetric part of the total stress, or total pressure $\psi$, concentration of first species $w_1$ and concentration of second species $w_2$ such that

$$
\begin{aligned}
-\mathbf{div}(2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}^s) - \psi\mathbf{I}) &= \rho\boldsymbol{b} + \mathbf{div}\,\boldsymbol{\sigma}_{\text{act}} && \text{in } \Omega \times (0, t_{\text{final}}], \\
\left(c_0 + \frac{\alpha^2}{\lambda}\right)\partial_t p^f - \frac{\alpha}{\lambda}\partial_t \psi - \frac{1}{\eta}\,\mathrm{div}(\kappa\nabla p^f) &= \ell && \text{in } \Omega \times (0, t_{\text{final}}], \\
\psi - \alpha p^f + \lambda\,\mathrm{div}\,\boldsymbol{u}^s &= 0 && \text{in } \Omega \times (0, t_{\text{final}}], \\
\partial_t w_1 + \boldsymbol{u}^s \cdot \nabla w_1 - \mathrm{div}(D_1(\boldsymbol{x})\,\nabla w_1) &= f(w_1, w_2, \boldsymbol{u}^s) && \text{in } \Omega \times (0, t_{\text{final}}], \\
\partial_t w_2 + \boldsymbol{u}^s \cdot \nabla w_2 - \mathrm{div}(D_2(\boldsymbol{x})\,\nabla w_2) &= g(w_1, w_2, \boldsymbol{u}^s) && \text{in } \Omega \times (0, t_{\text{final}}],
\end{aligned}
\tag{1.1}
$$

here $\kappa(\boldsymbol{x})$ is the hydraulic conductivity of the porous medium (possibly anisotropic), $\rho$ is the density of the solid material, $\eta$ is the constant viscosity of the interstitial fluid, $c_0$ is the constrained specific storage coefficient, $\alpha$ is the Biot-Willis consolidation parameter, and $\mu, \lambda$ are the shear and dilation moduli associated with the constitutive law of the solid structure, $D_1, D_2$ are positive definite diffusion matrices. Nevertheless, for sake of fixing ideas and to specify

the coupling effects also through a stability analysis, that is conducted in [15], they chose the coupling terms $f$ and $g$ as

$$f(w_1, w_2, \boldsymbol{u}^s) = \beta_1(\beta_2 - w_1 + w_1^2 w_2) + \gamma\, w_1 \text{ div } \boldsymbol{u}^s,$$
$$g(w_1, w_2, \boldsymbol{u}^s) = \beta_1(\beta_3 - w_1^2 w_2) + \gamma\, w_2 \text{ div } \boldsymbol{u}^s,$$

where $\beta_1, \beta_2, \beta_3, \gamma$ are positive model constants. The active component of total Cauchy stress, or active stress is given as $\boldsymbol{\sigma}_{\text{act}} = -\tau\, r(w_1, w_2)\boldsymbol{k} \otimes \boldsymbol{k}$, that is, $\boldsymbol{\sigma}_{\text{act}}$ operates primarily on a given constant direction $\boldsymbol{k}$, and its intensity depends on a scalar field $r(w_1, w_2)$ and on a positive constant $\tau$.

We endow the problem (1.1) with appropriate initial data

$$\boldsymbol{u}^s(0) = \boldsymbol{u}_0^s,\ p^f(0) = p_0^f,\ \psi(0) = \psi_0,\ w_i(0) = w_{i,0},\ \text{in } \Omega \times \{0\}, \qquad (1.2)$$

for $i = 1, 2$ and boundary conditions in the following manner

$$\boldsymbol{u}^s = \boldsymbol{0} \quad \text{and} \quad \frac{\kappa}{\eta}\nabla p^f \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma \times (0, t_{\text{final}}],$$

$$D_1(\boldsymbol{x})\nabla w_1 \cdot \boldsymbol{n} = 0 \quad \text{and} \quad D_2(\boldsymbol{x})\nabla w_2 \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma \times (0, t_{\text{final}}],$$

$$[2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}^s) - \psi\,\mathbf{I} + \boldsymbol{\sigma}_{\text{act}}]\boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad p^f = 0 \qquad \text{on } \Sigma \times (0, t_{\text{final}}],$$

$$w_1 = 0 \quad \text{and} \quad w_2 = 0 \qquad \text{on } \Sigma \times (0, t_{\text{final}}],$$

where the boundary $\partial\Omega = \Gamma \cup \Sigma$ is disjointly split into $\Gamma$ and $\Sigma$ where we prescribe clamped boundaries and zero fluid normal fluxes; and zero (total) traction together with constant fluid pressure, respectively. Moreover, zero concentrations normal fluxes are prescribed on $\partial\Omega$. We point out that, if we would like to start with a model in terms of the divergence ($\text{div}(w_i\boldsymbol{u}^s)$ instead of $\boldsymbol{u}^s \cdot \nabla w_i$ in (1.1), $i \in \{1, 2\}$), we need to assume zero total flux (including the advective term, see, e.g., [2]). Homogeneity of the boundary conditions is only assumed to simplify the exposition of the subsequent analysis.

This paper is structured as follows. In Section 2, we propose the weak formulation by introducing suitable Sobolev spaces. The fully discretized scheme is presented and proved that the scheme is well-posed in Section 3. We established *a-priori* error estimates in Section 4 with the help of Stokes and elliptic projection operators. In Section 5, we implement the numerical examples to verify the convergence results. Finally, we conclude the work by providing the future directions in Section 6.

## 2   Weak formulation

We will use the following notations for the Sobolev spaces in this article.

$$\mathbf{V} := [H_\Gamma^1(\Omega)]^2, \quad Q := H_\Sigma^1(\Omega), \quad Z := L^2(\Omega) \text{ and } W := H_\Gamma^1(\Omega).$$

Let us multiply (1.1) by adequate test functions and integrate by parts (in space) whenever appropriate. Incorporating the boundary conditions (1.3), the variational problem formulated as: For a given $t > 0$ and initial conditions (1.2), find $\boldsymbol{u}^s \in \mathbf{V}, p^f \in Q, \psi \in Z,\ w_1, w_2 \in W$ such that

$$a_1(\boldsymbol{u}^s, \boldsymbol{v}^s) + b_1(\boldsymbol{v}^s, \psi) = F_{\boldsymbol{b},r}(w_1, w_2; \boldsymbol{v}^s) \quad \forall \boldsymbol{v}^s \in \mathbf{V}, \qquad (2.1)$$

$$\tilde{a}_2(\partial_t p^f, q^f) + a_2(p^f, q^f) - b_2(q^f, \partial_t \psi) = G_\ell(q^f) \quad \forall q^f \in Q,$$
$$b_1(\boldsymbol{u}^s, \phi) + b_2(p^f, \phi) - a_3(\psi, \phi) = 0 \quad \forall \phi \in Z,$$
$$m(\partial_t w_1, s) + a_4(w_1, s) + c(\boldsymbol{u}^s; w_1, s) = J_f(w_1, w_2, \boldsymbol{u}^s; s) \quad \forall s \in W, \qquad (2.2)$$
$$m(\partial_t w_2, s) + a_5(w_2, s) + c(\boldsymbol{u}^s; w_2, s) = J_g(w_1, w_2, \boldsymbol{u}^s; s) \quad \forall s \in W, \qquad (2.3)$$

where the formulation with the bilinear forms $a_1 : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$, $a_2 : Q \times Q \to \mathbb{R}$, $a_3 : Z \times Z \to \mathbb{R}$, $a_4, a_5 : W \times W \to \mathbb{R}$, $b_1 : \mathbf{V} \times Q \to \mathbb{R}$, $b_2 : Q \times Z \to \mathbb{R}$, the trilinear form $c : \mathbf{V} \times W \times W \to \mathbb{R}$, and linear functionals $F_{\boldsymbol{b},r} : \mathbf{V} \to \mathbb{R}$ (for $r$ known), $G_\ell : Q \to \mathbb{R}$, $J_f, J_g : W \to \mathbb{R}$ (for known $f$ and known $g$), are defined as

$$a_1(\boldsymbol{u}^s, \boldsymbol{v}^s) := \mu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}^s) : \boldsymbol{\varepsilon}(\boldsymbol{v}^s), \quad b_1(\boldsymbol{v}^s, \phi) := -\int_\Omega \phi \operatorname{div} \boldsymbol{v}^s,$$

$$\tilde{a}_2(p^f, q^f) := \left( c_0 + \frac{\alpha^2}{\lambda} \right) \int_\Omega p^f q^f, \quad a_2(p^f, q^f) := \frac{1}{\eta} \int_\Omega \kappa(\boldsymbol{x}) \nabla p^f \cdot \nabla q^f,$$

$$b_2(p^f, \phi) := \frac{\alpha}{\lambda} \int_\Omega p^f \phi, \quad a_3(\psi, \phi) := \frac{1}{\lambda} \int_\Omega \psi \phi,$$

$$m(w_i, s) := \int_\Omega w_i s, \quad a_{3+i}(w_i, s) := \int_\Omega D_i(\boldsymbol{x}) \nabla w_i \cdot \nabla s, \text{ for } i = 1, 2,$$

$$c(\boldsymbol{u}^s; w, s) := \int_\Omega (\boldsymbol{u}^s \cdot \nabla w) s, \quad F_{\boldsymbol{b},r}(w_1, w_2; \boldsymbol{v}^s) := F_{\boldsymbol{b}}(\boldsymbol{v}^s) + F_r(w_1, w_2; \boldsymbol{v}^s)$$

where

$$F_{\boldsymbol{b}}(\boldsymbol{v}^s) := \rho \int_\Omega \boldsymbol{b} \cdot \boldsymbol{v}^s, \quad F_r(w_1, w_2; \boldsymbol{v}^s) := \tau \int_\Omega r(w_1, w_2) \boldsymbol{k} \otimes \boldsymbol{k} : \boldsymbol{\varepsilon}(\boldsymbol{v}^s),$$

$$G_\ell(q^f) := \int_\Omega \ell \, q^f, \quad J_z(w_1, w_2, \boldsymbol{u}^s; s) := \int_\Omega z(w_1, w_2, \boldsymbol{u}^s) \, s, \quad z = f, g.$$

*Remark 1.* Throughout this paper, $z$ stands for the coupling terms $f$ and $g$.

We will consider that the initial data (1.2) are non-negative and regular enough. Moreover, throughout the text we will assume that the anisotropic permeability $\kappa(\boldsymbol{x})$ and the diffusion matrices $D_1(\boldsymbol{x}), D_2(\boldsymbol{x})$ are uniformly bounded and positive definite in $\Omega$. The latter means that, there exist positive constants $\kappa_1, \kappa_2$, and $D_i^a, D_i^b$, $i \in \{1, 2\}$, such that $\forall \boldsymbol{w} \in \mathbb{R}^d$, $d = 1, 2$, $\forall \boldsymbol{x} \in \Omega$,

$$\kappa_1 |\boldsymbol{w}|^2 \le \boldsymbol{w}^{\mathrm{t}} \kappa(\boldsymbol{x}) \boldsymbol{w} \le \kappa_2 |\boldsymbol{w}|^2 \quad \text{and} \quad D_i^a |\boldsymbol{w}|^2 \le \boldsymbol{w}^{\mathrm{t}} D_i(\boldsymbol{x}) \boldsymbol{w} \le D_i^b |\boldsymbol{w}|^2.$$

Also, for a fixed $\boldsymbol{u}^s$, the reaction kinetics $f(w_1, w_2, \cdot), g(w_1, w_2, \cdot)$ satisfy the growth conditions, that is,

$$|z(w_1, w_2, \cdot) - z(\tilde{w}_1, \tilde{w}_2, \cdot)| \le C(|w_1 - \tilde{w}_1| + |w_2 - \tilde{w}_2|)$$
$$|z(w_1, w_2, \cdot)| \le C(1 + |w_1| + |w_2|),$$

for given $w_1, w_2 \in \mathbb{R}$, the scalar field $r(w_1, w_2)$ and reaction kinetics $f(\cdot, \cdot, \boldsymbol{u}^s)$, $g(\cdot, \cdot, \boldsymbol{u}^s)$ such that

$$|r(w_1, w_2) - r(\tilde{w}_1, \tilde{w}_2)| \le C(|w_1 - \tilde{w}_1| + |w_2 - \tilde{w}_2|),$$
$$|r(w_1, w_2)| \le |w_1| + |w_2|, \ |z(\cdot, \cdot, \boldsymbol{u}^s) - z(\cdot, \cdot, \tilde{\boldsymbol{u}}^s)| \le C|\operatorname{div} \boldsymbol{u}^s - \operatorname{div} \tilde{\boldsymbol{u}}^s|,$$

for fixed $w_1, w_2$.

In addition, the terms in (2.1) fulfill the continuity bounds, coercivity and positivity bounds, and $C_{k,1}$ and $C_{k,2}$ are the positive constants satisfying

$$C_{k,1}\|\boldsymbol{v}^s\|_{1,\Omega}^2 \le \|\boldsymbol{\varepsilon}(\boldsymbol{v}^s)\|_{0,\Omega}^2 \le C_{k,2}\|\boldsymbol{v}^s\|_{1,\Omega}^2$$

and $c_p$ is the Poincaré constant. Moreover, the bilinear form $b_1$ satisfies the inf-sup condition (see, e.g., [17]).

## 3 Discrete formulations and wellposedness

In this section, by following virtual element methods for space and Euler backward scheme for time discretizations, we present a fully discrete scheme corresponding to (2.1). We also address the stability, existence and uniqueness of solution for the discrete problem.

### 3.1 Virtual element discretizations

Let the domain $\Omega$ be discretised into the family of the polygonal meshes $\mathcal{T}_h$ with mesh size $h$ and elements $K$, vertices on each element $K$ as $V_i$, $1 \le i \le N_K^v$ with $N_K^v$ number of vertices in $K$, and any edge in the polygonal mesh is denoted by $e$. For any natural number $k$, let $\mathbb{P}_k(S)$ and $\mathcal{M}_k(S)$ represent the space of polynomials and monomials, or scaled polynomials of degree less than or equal to $k$ for any $S \subset \mathbb{R}^2$. We will write $a \lesssim b$ for the expressions $a \le Cb$, where $C$ is a generic constant independent of parameters $h$ and $\Delta t$.

We also suppose that the polygonal mesh satisfy the assumptions (refer [5,8]).

Before proceeding to define the virtual element spaces, we will take few notations as, for any smooth enough $u, v$,

$$(u,v)_{\nabla,K} := (\nabla u, \nabla v)_{0,K}, \quad (u,v)_{\varepsilon,K} := (\boldsymbol{\varepsilon}(u), \boldsymbol{\varepsilon}(v))_{0,K}.$$

The standard energy projection $\Pi_K^\nabla : H^1(K) \to \mathbb{P}_1(K)$ is defined as

$$(\Pi_K^\nabla q - q, p_1)_{\nabla,K} = 0 \quad \forall p_1 \in \mathbb{P}_1(K).$$

Note that the above definition on the $H^1$ scalar product is defined up to a constant, and in order to determine the constant, we define another projection $P_K^0$ as follows

$$P_K^0(\Pi_K^\nabla q - q) = 0, \quad \text{where} \quad P_K^0 q := \frac{1}{N_K^v}\sum_{i=1}^{N_K^v} q(V_i).$$

The vectorial energy projection from the vector space $[H^1(K)]^2$ to $[\mathbb{P}_1(K)]^2$ is denoted as $\boldsymbol{\Pi}_K^\nabla$. A variant of the vectorial projection $\boldsymbol{\Pi}_K^\nabla$ and supported by the bilinear form $a_1^K(\cdot,\cdot)$, we define a projection $\boldsymbol{\Pi}_K^\varepsilon$ as,

$$(\boldsymbol{\Pi}_K^\varepsilon \boldsymbol{v} - \boldsymbol{v}, \boldsymbol{p}_1)_{\varepsilon,K} = 0 \quad \forall \boldsymbol{p}_1 \in [\mathbb{P}_1(K)]^2, \ \boldsymbol{v} \in [H^1(K)]^2$$

and the function $\boldsymbol{p}_1 \in \ker(a_1^K(\cdot,\cdot))$ are again taken from

$$\mathbf{P}_K^0(\boldsymbol{\Pi}_K^\epsilon \boldsymbol{v} - \boldsymbol{v}, \boldsymbol{p}_1) := \frac{1}{N_K^v} \sum_{i=1}^{N_K^v} (\boldsymbol{\Pi}_K^\epsilon \boldsymbol{v} - \boldsymbol{v})(V_i) \cdot \boldsymbol{p}_1(V_i) = 0.$$

The classical $L^2$-projection operator $\Pi_K^0 : L^2(K) \to \mathbb{P}_1(K)$ is expressed as

$$(\Pi_K^0 v - v, p_1)_{0,K} = 0 \quad \forall v \in L^2(K), \ p_1 \in \mathbb{P}_1(K).$$

Similar to the energy projections, the projections $\boldsymbol{\Pi}_K^{0,0} : [L^2(K)]^2 \to [\mathbb{P}_0(K)]^2$ and $\boldsymbol{\Pi}_K^0 : [L^2(K)]^2 \to [\mathbb{P}_1(K)]^2$ are identified as the vectorial $L^2$ projection onto constants and linear polynomials, respectively. We stress that these operators not only help us in deriving the optimal error estimates but are also useful in the computation of discrete bilinear forms (to be defined later).

Then the local VE spaces are introduced as follows [1, 4]:

$$\mathbf{V}_h(K) := \left\{ \boldsymbol{v}^s \in \mathcal{B}_{\boldsymbol{v}}(\partial K) : \begin{cases} -\Delta \boldsymbol{v}^s + \nabla s = 0 \text{ in } K, \text{ for some } s \in L^2(K), \\ \operatorname{div} \boldsymbol{v}^s \in \mathbb{P}_0(K) \end{cases} \right\},$$

$$Q_h(K) := \left\{ q^f \in \mathcal{B}_q(\partial K) : \begin{cases} -\Delta q^f = 0 \text{ in } K, \\ (q^f - \Pi_K^\nabla q^f, m_\alpha)_{0,K} = 0 \quad \forall m_\alpha \in \mathcal{M}_1(K) \end{cases} \right\},$$

$$Z_h(K) := \mathbb{P}_0(K),$$

where the boundary space is given by

$$\mathcal{B}_{\boldsymbol{v}}(\partial K) := \left\{ \boldsymbol{v}^s \in [C^0(\partial K) \cap H^1(K)]^2 : \begin{cases} (\boldsymbol{v}^s \cdot \boldsymbol{t}_K^e)|_e \in \mathbb{P}_1(e), \\ (\boldsymbol{v}^s \cdot \boldsymbol{n}_K^e)|_e \in \mathbb{P}_2(e) \end{cases} \quad \forall e \in \partial K \right\},$$

$$\mathcal{B}_q(\partial K) := \{ q^f \in C^0(\partial K) \cap H^1(K) : q^f|_e \in \mathbb{P}_1(e) \quad \forall e \in \partial K \}.$$

The approximation space for displacement is $\mathbf{V}_h$, for total pressure, $Z_h$, and for pressure and concentration is $Q_h(K)$. We have the following degrees of freedom depending on the corresponding spaces (refer [1, 4, 29] for details on unisolvance) as, for displacement: the values at all vertices of the element $K$, and value of $\boldsymbol{v}^s \cdot \boldsymbol{n}_K^e$ at mid point on each edge $e \in \partial K$; for total volumetric stress: value of $\psi$ at any point in $K$; and for pressure, or concentration: the values of $q^f$ at vertices of the element $K$.

Then any global space $X_h$ corresponding to space $X$ (choices are $X = \mathbf{V}, Q, W$) is given as

$$X_h := \{ v \in X : v|_K \in X_h(K) \quad \forall K \in \mathcal{T}_h \}.$$

Note from above that the local discrete spaces and its degrees of freedom for pressure and concentration are the same, and thus have the same local projection operators on each element $K \in \mathcal{T}_h$.

We define the local discrete bilinear forms by considering the computability,

consistency and stability, as:

$$a_1^{h,K}(\boldsymbol{u}_h^s, \boldsymbol{v}_h^s) := a_1^K(\boldsymbol{\Pi}_K^{\boldsymbol{\varepsilon}} \boldsymbol{u}_h^s, \boldsymbol{\Pi}_K^{\boldsymbol{\varepsilon}} \boldsymbol{v}_h^s) + 2\mu\, \mathcal{S}^{\boldsymbol{\varepsilon},K}((I - \boldsymbol{\Pi}_K^{\boldsymbol{\varepsilon}})\boldsymbol{u}_h^s, (I - \boldsymbol{\Pi}_K^{\boldsymbol{\varepsilon}})\boldsymbol{v}_h^s),$$

$$b_1^K(\boldsymbol{v}_h^s, \phi_h) = (\operatorname{div}\boldsymbol{v}_h^s, \phi_h)_{0,K}, \quad b_2^K(q_h^f, \phi_h) := \alpha\,\lambda^{-1}(\Pi_K^0 q_h^f, \phi_h)_{0,K},$$

$$\tilde{a}_2^{h,K}(p_h^f, q_h^f) := \tilde{a}_2^K(\Pi_K^0 p_h^f, \Pi_K^0 q_h^f) + \left(c_0 + \frac{\alpha^2}{\lambda}\right) \mathcal{S}^{0,K}((I - \Pi_K^0)p_h^f, (I - \Pi_K^0)q_h^f),$$

$$a_2^{h,K}(p_h^f, q_h^f) := a_2^K(\Pi_K^{\nabla} p_h^f, \Pi_K^{\nabla} q_h^f) + \bar{\kappa}\,\eta^{-1}\, \mathcal{S}^{\nabla,K}((I - \Pi_K^{\nabla})p_h^f, (I - \Pi_K^{\nabla})q_h^f),$$

$$a_3^K(\psi_h, \phi_h) := (\psi_h, \phi_h)_{0,K}, \quad c^{h,K}(\boldsymbol{v}_h^s; w_h, s_h)$$

$$:= \left((\boldsymbol{\Pi}_K^0 \boldsymbol{v}_h^s) \cdot (\boldsymbol{\Pi}_K^{0,0} \nabla w_h), \Pi_K^0 s_h\right)_{0,K},$$

$$a_i^{h,K}(w_h, s_h) := a_i^K(\Pi_K^{\nabla} w_h, \Pi_K^{\nabla} s_h)$$

$$+ \bar{D}_{i-3}\, \mathcal{S}^{\nabla,K}((I - \Pi_K^{\nabla})w_h, (I - \Pi_K^{\nabla})s_h),\ i = 4,5,$$

$$m^{h,K}(w_h, s_h) := m^K(\Pi_K^0 w_h, \Pi_K^0 s_h) + \mathcal{S}^{0,K}((I - \Pi_K^0)w_h, (I - \Pi_K^0)s_h),$$

where $\bar{\kappa}$ and $\bar{D}_i$'s denoting the average values of the respective parameters, and the stabilization terms on each $K$, with Ndof denoting the dimension of the associated space to the variables (for instance, we have Ndof as dimension of $Q_h(K)$ for the variable $p_h, q_h$), as

$$\mathcal{S}^{\nabla,K}(w_h,\ s_h) := \sum_{r=1}^{\text{Ndof}} \operatorname{dof}_r(w_h)\, \operatorname{dof}_r(s_h) \text{ with } \Pi_K^{\nabla} w_h, \Pi_K^{\nabla} s_h = 0,$$

$$\mathcal{S}^{0,K}(w_h,\ s_h) := \operatorname{area}(K) \sum_{r=1}^{\text{Ndof}} \operatorname{dof}_r(w_h)\, \operatorname{dof}_r(s_h) \text{ with } \Pi_K^0 w_h, \Pi_K^0 s_h = 0,$$

$$\mathcal{S}^{\boldsymbol{\varepsilon},K}(\boldsymbol{u}_h^s, \boldsymbol{v}_h^s) := \sum_{r=1}^{\text{Ndof}} \operatorname{dof}_r(\boldsymbol{u}_h^s)\, \operatorname{dof}_r(\boldsymbol{v}_h^s) \text{ with } \Pi_K^{\boldsymbol{\varepsilon}} \boldsymbol{u}_h^s, \Pi_K^{\boldsymbol{\varepsilon}} \boldsymbol{v}_h^s = 0.$$

Then the global discrete bilinear forms and discrete functional are defined naturally for any discrete form $a^h(\cdot, \cdot) : X_h \times X_h \to \mathbb{R}$ as

$$a^h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} a^{h,K}(u_h, v_h) \quad \forall u_h, v_h \in X_h.$$

The discrete functionals are defined in terms of $L^2$ projections as

$$G_l^h(q_h)|_K := \langle \ell_h, q_h \rangle_{0,K}, \quad F_{\boldsymbol{b},r}^h(w_{1,h}, w_{2,h}; \boldsymbol{v}_h^s) := F_{\boldsymbol{b}}^h(\boldsymbol{v}_h^s) + F_r^h(w_{1,h}, w_{2,h}; \boldsymbol{v}_h^s),$$

$$F_{\boldsymbol{b}}^h(\boldsymbol{v}_h^s)|_K := \rho\,\langle \boldsymbol{b}_h, \boldsymbol{v}_h^s \rangle_{0,K},$$

$$F_r^h(w_{1,h}, w_{2,h}; \boldsymbol{v}_h^s)|_K := \tau\,\langle r_h(w_{1,h}, w_{2,h})(\boldsymbol{k} \otimes \boldsymbol{k}), \boldsymbol{\varepsilon}(\boldsymbol{v}_h^s) \rangle_{0,K},$$

$$J_z^h(w_{1,h}, w_{2,h}, \boldsymbol{u}_h^s; s_h)|_K := \langle z_h(w_{1,h}, w_{2,h}, \boldsymbol{u}_h^s), s_h \rangle_{0,K},$$

where

$$\boldsymbol{b}_h|_K := \boldsymbol{\Pi}_K^{0,0} \boldsymbol{b}, \quad r_h(w_{1,h}, w_{2,h})|_K := \Pi_K^{0,0} r\left(\Pi_K^0 w_{1,h}, \Pi_K^0 w_{2,h}\right),$$

$$\ell_h|_K := \Pi_K^0 \ell, \quad z_h(w_{1,h}, w_{2,h}, \boldsymbol{u}_h^s)|_K := \Pi_K^0 z\left(\Pi_K^0 w_{1,h}, \Pi_K^0 w_{2,h}, \boldsymbol{u}_h^s\right).$$

The stabilization terms satisfies the stability condition with respect to the norm associated with the respective bilinear forms, that is, for $\boldsymbol{v}_h^s \in \mathbf{V}_h(K)$, $s_h \in W_h(K)$, or $Q_h(K)$, we have

$$c_* |s_h|_{1,K}^2 \le \mathcal{S}^{\nabla,K}(s_h, \ s_h) \le c^* |s_h|_{1,K}^2,$$
$$\tilde{c}_* \|s_h\|_{0,K}^2 \le \mathcal{S}^{0,K}(s_h, \ s_h) \le \tilde{c}^* \|s_h\|_{0,K}^2,$$
$$\hat{c}_* \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h^s)\|_{0,K}^2 \le \mathcal{S}^{\boldsymbol{\varepsilon},K}(\boldsymbol{v}_h^s, \ \boldsymbol{v}_h^s) \le \hat{c}^* \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h^s)\|_{0,K}^2,$$

where $c_*, c^*, \tilde{c}_*, \tilde{c}^*, \hat{c}_*, \hat{c}^*$ are constants independent of $h_K$. With the help of the stability of the projection operators and stabilizations terms, we obtain the following lemma ( for details, see [6, 9, 12] and references within).

**Lemma 1.** *For all* $\boldsymbol{u}_h^s, \boldsymbol{v}_h^s \in \mathbf{V}_h$, $p_h^f, q_h^f \in Q_h$, $w_h, s_h \in W_h$, *we have, for* $i = 1, 2$,

$$a_1^h(\boldsymbol{u}_h^s, \boldsymbol{v}_h^s) \le 2\mu\hat{\alpha}^* \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^s)\|_{0,\Omega} \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h^s)\|_{0,\Omega},$$
$$\tilde{a}_2^h(p_h^f, q_h^f) \le \tilde{\alpha}^* \left(c_0 + \frac{\alpha^2}{\lambda}\right) \left\|p_h^f\right\|_{0,\Omega} \left\|q_h^f\right\|_{0,\Omega},$$
$$a_2^h(p_h^f, q_h^f) \le \alpha^* \kappa_2 \eta^{-1} \left\|\nabla p_h^f\right\|_{0,\Omega} \left\|\nabla q_h^f\right\|_{0,\Omega},$$
$$a_{3+i}^h(w_h, s_h) \le \alpha^* D_i^b \|\nabla w_h\|_{0,\Omega} \|\nabla s_h\|_{0,\Omega},$$
$$F_{\boldsymbol{b},r}^h(w_{1,h}, w_{2,h}; \boldsymbol{v}_h^s) \lesssim (\rho \|\boldsymbol{b}\|_{0,\Omega} \|\boldsymbol{v}_h^s\|_{0,\Omega} + \tau \sqrt{C_{k,2}} \|r_h\|_{0,\Omega} \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h^s)\|_{0,\Omega}),$$
$$m^h(w_h, s_h) \le \|w_h\|_{0,\Omega} \|s_h\|_{0,\Omega}, \quad G_\ell^h(q_h^f) \le \|\ell\|_{0,\Omega} \left\|q_h^f\right\|_{0,\Omega},$$
$$c^h(\boldsymbol{v}_h^s; w_h, s_h) \lesssim |\boldsymbol{v}_h^s|_{1,\Omega} \|w_h\|_{1,\Omega} \|s_h\|_{1,\Omega},$$
$$J_z^h(w_{1,h}, w_{2,h}, \boldsymbol{u}_h^s; s_h) \le \|z_h\|_{0,\Omega} \|s_h\|_{0,\Omega}. \tag{3.1}$$

Moreover, the coercivity properties can be obtained through similar arguments, which implies that for all $\boldsymbol{v}_h^s \in \mathbf{V}_h$, $q_h^f \in Q_h$, $s_h \in W_h$,

$$a_1^h(\boldsymbol{v}_h^s, \boldsymbol{v}_h^s) \ge 2\mu\hat{\alpha}_* \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h^s)\|_{0,\Omega}^2, \quad a_2^h(q_h^f, q_h^f) \ge \alpha_* \kappa_1 \eta^{-1} \left\|\nabla q_h^f\right\|_{0,\Omega}^2,$$
$$m^h(s_h, s_h) \ge \tilde{\alpha}_* \|s_h\|_{0,\Omega}^2, \quad a_{3+i}^h(s_h, s_h) \ge \alpha_* D_i^a \|\nabla s_h\|_{0,\Omega}^2 \quad \forall i = 1, 2.$$

Also, the discrete inf-sup condition hold on $(\mathbf{V}_h, Z_h)$ (refer [7, 8, 29]).

By introducing the adequate discrete spaces associated with velocity, pressure, total volumetric stress, and concentrations, the fully discrete formulation is described in the next subsection.

## 3.2 Fully discrete scheme

Let us discretise the time interval $(0, t_{\text{final}}]$ into $N$ equispaced points and time step $\Delta t$ with denoting $n^{\text{th}}$ time step as $t^n = n\Delta t$ and $n = 1, \dots, N$, and use the following general notation for the first order backward difference $\Delta t \, \delta_t X^n := X^n - X^{n-1}$. In this way, we can write a discrete form of (2.1): From the given initial data $\boldsymbol{u}_h^{s,0}, p_h^{f,0}, \psi_h^0, w_{1,h}^0, w_{2,h}^0$ (which will be projections of the continuous

initial conditions of each field) and starting with $n = 1$, we first solve for $\boldsymbol{u}_h^{s,n} \in \mathbf{V}_h$, $p_h^{f,n} \in Q_h$, $\psi_h^n \in Z_h$ such that, for all $\boldsymbol{v}_h^s \in \mathbf{V}_h$, $q_h^f \in Q_h$, $\phi_h \in Z_h$,

$$a_1^h(\boldsymbol{u}_h^{s,n}, \boldsymbol{v}_h^s) + b_1(\boldsymbol{v}_h^s, \psi_h^n) = F_{\boldsymbol{b},r}^{h,n}(w_{1,h}^{n-1}, w_{2,h}^{n-1}; \boldsymbol{v}_h^s), \tag{3.2a}$$

$$\tilde{a}_2^h(\delta_t p_h^{f,n}, q_h^f) + a_2^h(p_h^{f,n}, q_h^f) - b_2(q_h^f, \delta_t \psi_h^n) = G_\ell^{h,n}(q_h^f), \tag{3.2b}$$

$$b_1(\boldsymbol{u}_h^{s,n}, \phi_h) + b_2(p_h^{f,n}, \phi_h) - a_3(\psi_h^n, \phi_h) = 0. \tag{3.2c}$$

Then we seek the concentrations $w_{1,h}^n, w_{2,h}^n \in W_h$ for given displacement $\boldsymbol{u}_h^{s,n} \in \mathbf{V}_h$ (solution of (3.2)) and known initial data $w_{1,h}^0, w_{2,h}^0$ such that $\forall s_h \in W_h$,

$$m^h(\delta_t w_{1,h}^n, s_h) + a_4^h(w_{1,h}^n, s_h) + c^h(\boldsymbol{u}_h^{s,n}; w_{1,h}^n, s_h) = J_f^{h,n}(w_{1,h}^{n-1}, w_{2,h}^{n-1}, \boldsymbol{u}_h^{s,n}; s_h), \tag{3.3a}$$

$$m^h(\delta_t w_{2,h}^n, s_h) + a_5^h(w_{2,h}^n, s_h) + c^h(\boldsymbol{u}_h^{s,n}; w_{2,h}^n, s_h) = J_g^{h,n}(w_{1,h}^{n-1}, w_{2,h}^{n-1}, \boldsymbol{u}_h^{s,n}; s_h). \tag{3.3b}$$

The above process of solving the problem continues iteratively for $n = 2, \ldots, N$.

We note that the above systems of equations are linear for each $n$ since we have considered the explicit scheme in time discretization.

### Existence and Uniqueness

We will show the well-posedness of the discrete scheme through stability and then uniqueness of the linear system of equations. We start it by introducing the discrete-in-time $l^2-$ norm as follows,

$$\|X\|_{l^2(V)}^2 := \|X\|_{l^2(0,t_n;V)}^2 = \Delta t \sum_{j=0}^n \|X^j\|_V^2.$$

Now, we collect the following important results for the further analysis:

$$\int_\Omega X^n \delta_t X^n = \frac{1}{2}\delta_t \|X^n\|_{0,\Omega}^2 + \frac{1}{2}\Delta t \|\delta_t X^n\|_{0,\Omega}^2, \tag{3.4}$$

$$\int_\Omega (\boldsymbol{v}^s \cdot \nabla w)w\, dx = \frac{1}{2}\int_\Omega \boldsymbol{v}^s \cdot \nabla w^2\, dx = -\frac{1}{2}\int_\Omega \operatorname{div}(\boldsymbol{v}^s)w^2\, dx,$$

for $\boldsymbol{v}^s \in \mathbf{V}$ and $w \in W$.

The next lemma is referred from our previous work [12] and it is needed for the analysis.

**Lemma 2.** *We have the following bound, for all $q_h^{f,n} \in Q_h$ and $\phi_h^n \in Z_h$ at each $n = 1, \ldots, N$*

$$(\Delta t)\sum_{j=1}^n L_h^j \geq \frac{1}{2}\sum_{K \in \mathcal{T}_h}\sum_{j=1}^n \delta_t \left( \frac{1}{\lambda} \left\| \alpha \Pi_K^0 q_h^{f,j} - \phi_h^j \right\|_{0,K}^2 + c_0 \left\| \Pi_K^0 q_h^{f,j} \right\|_{0,K}^2 \right.$$
$$\left. + \tilde{\alpha}_* \left( c_0 + \frac{\alpha^2}{\lambda} \right) \left\| (I - \Pi_K^0)q_h^{f,j} \right\|_{0,K}^2 \right),$$

*where*

$$L_h^n := \tilde{a}_2^h(\delta_t q_h^{f,n}, q_h^{f,n}) - b_2(q_h^{f,n}, \delta_t \phi_h^n) - b_2(\delta_t q_h^{f,n}, q_h^{f,n}) + a_3(\delta_t \phi_h^n, \phi_h^n).$$

Next, we recall a well known lemma utilized to handle analysis of the non-linear and time dependent problems. Also, the proof of the following lemma can be consulted from [18, Lemma 5.1].

**Lemma 3 [Generalised Discrete Gronwall lemma].** *Let* $k$, $B$, *and* $a_j, b_j$, $c_j, \gamma_j$ *for integer* $j \geq 0$ *be non-negative numbers such that, for* $n > 0$, *we have*

$$a_n + k\sum_{j=0}^{n} b_j \leq k\sum_{j=0}^{n} \gamma_j a_j + k\sum_{j=0}^{n} c_j + B.$$

*If* $k\gamma_j < 1$, *then*

$$a_n + k\sum_{j=0}^{n} b_j \leq \exp\left(k\sum_{j=0}^{n} (1 - k\gamma_j)^{-1}\gamma_j\right)\left\{k\sum_{j=0}^{n} c_j + B\right\}. \qquad (3.5)$$

**Theorem 1 [Existence and Uniqueness].** *For each* $n = 1, \ldots, N$, *the fully-discrete formulation* (3.2)–(3.3) *of the coupled problem* (1.1) *has a unique solution* $(\boldsymbol{u}_h^{s,n}, p_h^{f,n}, \psi_h^n, w_{1,h}^n, w_{2,h}^n) \in \mathbf{V}_h \times Q_h \times Z_h \times W_h \times W_h$.

*Proof.* The linear problem (3.2) in the form of an uncoupled fully discrete scheme for poroelasticity problem is well-posed for a given data and can be referred from [12]. We will ensure the existence of a unique solution of linear uncoupled ADR equation (3.3) by virtue of the Lax-Milgram lemma. In order to proceed, we define the bilinear forms for each $i = 1, 2$ and given $\boldsymbol{u}_h^{s,n}$ as solution of the problem (3.2a)–(3.2c) as,

$$\mathcal{C}_i^h(w_{i,h}^n, s_h) := m^h(w_{i,h}^n, s_h) + \Delta t\Big(a_{3+i}^h(w_{i,h}^n, s_h) + c^h(\boldsymbol{u}_h^{s,n}; w_{i,h}^n, s_h)\Big).$$

We can rewrite the uncoupled ADR problem (3.3a)–(3.3b) for all $s_h \in W_h$ as

$$m^h(w_{i,h}^n, s_h) + \Delta t\Big(a_{3+i}^h(w_{i,h}^n, s_h) + c^h(\boldsymbol{u}_h^{s,n}; w_{i,h}^n, s_h)\Big)$$
$$= \Delta t\, J_z^{h,n}(w_{1,h}^{n-1}, w_{2,h}^{n-1}, \boldsymbol{u}_h^{s,n}; s_h) + m^h(w_{i,h}^{n-1}, s_h),$$

and $i = 1, 2$. The continuity of right hand side obtained using the bounds of the linear functionals $m^h(w_{1,h}^{n-1}, \cdot)$ and $J_z^{h,n}(w_{1,h}^{n-1}, w_{2,h}^{n-1}, \boldsymbol{u}_h^{s,n}; \cdot)$. Now, we will prove the coercivity of bilinear form $\mathcal{C}_i^h(\cdot, \cdot)$. For all $s_h \in W_h$, the usage of Inverse inequality for polynomials leads to

$$c_h^K(\boldsymbol{u}_h^{s,n}; s_h, s_h) \leq C\|\boldsymbol{u}_h^{s,n}\|_{\infty,K} \|\nabla s_h\|_{0,K} \|s_h\|_{0,K}.$$

Now, for any $s_h \in W_h$, the use of above bound for $c_h^K(\boldsymbol{u}_h^{s,n}; \cdot, \cdot)$, coercivity of bilinear forms $m^h(\cdot, \cdot)$ and $a_{3+i}^h(\cdot, \cdot)$, and Young's inequality, we get

$$\mathcal{C}_i^h(s_h, s_h) = m^h(s_h, s_h) + \Delta t\Big(a_{3+i}^h(s_h, s_h) + c^h(\boldsymbol{u}_h^{s,n}; s_h, s_h)\Big)$$
$$\geq \left(\tilde{\alpha}_* - \frac{C\Delta t}{2\alpha_* D_i^a}\|\boldsymbol{u}_h^{s,n}\|_{\infty,\Omega}^2\right)\|s_h\|_{0,\Omega}^2 + \frac{\alpha_*}{2}D_i^a\Delta t\|\nabla s_h\|_{0,\Omega}^2.$$

Choosing $\Delta t > 0$ small enough and require $D_i^a$ so that $\tilde{\alpha}_* \geq \frac{C\|\boldsymbol{u}_h^{s,n}\|_{\infty,\Omega}^2}{2\alpha_* D_i^a}\Delta t$, we achieve the coercivity of $\mathcal{C}_i^h(\cdot,\cdot)$. Also, the continuity of the bilinear form is followed from the boundedness of the discrete bilinear forms (3.1). Thus, the Lax-Milgram lemma deduces the existence of a unique solution. $\square$

**Theorem 2 [Stability of the fully discrete scheme].** *Assume that the solution $(\boldsymbol{u}_h^{s,n}, p_h^{f,n}, \psi_h^n, w_{1,h}^n, w_{2,h}^n) \in \mathbf{V}_h \times Q_h \times Z_h \times W_h \times W_h$ of fully discrete scheme (3.2)–(3.3) satisfies, for constant $C > 0$ (independent of $h$ and $\Delta t$),*

$$\|\varepsilon(\boldsymbol{u}_h^{s,n})\|_{0,\Omega}^2 + \left\|p_h^f\right\|_{l^2(H^1(\Omega))}^2 + \|\psi_h^n\|_{0,\Omega}^2 + \sum_{i=1}^{2}\left(\|w_{i,h}^n\|_{0,\Omega}^2 + D_i^a\|w_{i,h}\|_{l^2(H^1(\Omega))}^2\right)$$

$$\leq C\Bigg(\left\|\varepsilon(\boldsymbol{u}_h^{s,0})\right\|_{0,\Omega}^2 + \left\|p_h^{f,0}\right\|_{0,\Omega}^2 + \|\psi_h^0\|_{0,\Omega}^2 + \sum_{i=1}^{2}\|w_{i,h}^0\|_{0,\Omega}^2 + \|\ell\|_{l^2(L^2(\Omega))}^2$$

$$+ \sum_{j=1}^{n}\left(1 + \|\boldsymbol{b}^j\|_{0,\Omega}^2\right)\Bigg).$$

*Proof.* Taking $\boldsymbol{v}_h^{s,n} = \delta_t\boldsymbol{u}_h^{s,n}$ in (3.2a), $q_h^f = p_h^{f,n}$ in (3.2b), and for time step $n$ and $n-1$ in Equation (3.2c) with $\phi_h = \psi_h^n$, and adding these equations gives

$$a_1^h(\boldsymbol{u}_h^{s,n}, \delta_t\boldsymbol{u}_h^{s,n}) + a_2^h(p_h^{f,n}, p_h^{f,n}) + \tilde{a}_2^h(\delta_t p_h^{f,n}, p_h^{f,n}) - b_2(p_h^{f,n}, \delta_t\psi_h^n)$$
$$- b_2(\delta_t p_h^{f,n}, \psi_h^n) + a_3(\delta_t\psi_h^n, \psi_h^n) = F_{\boldsymbol{b},r}^{h,n}(w_{1,h}^{n-1}, w_{2,h}^{n-1}; \delta_t\boldsymbol{u}_h^{s,n}) + G_\ell^{h,n}(p_h^{f,n}).$$

Summing over $n$, and a use of (3.4) and bound of $L_h^n$ from Lemma (2), we get

$$\frac{\mu}{2}\|\varepsilon(\boldsymbol{u}_h^{s,n})\|_{0,\Omega}^2 + \mu\frac{\Delta t^2}{2}\sum_{j=1}^{n}\left\|\delta_t\varepsilon(\boldsymbol{u}_h^{s,j})\right\|_{0,\Omega}^2 + \frac{\kappa_1}{\eta}\Delta t\sum_{j=1}^{n}\left\|\nabla p_h^{f,j}\right\|_{0,\Omega}^2$$

$$+ \frac{1}{2}\sum_{K\in\mathcal{T}_h}\left(\frac{1}{\lambda}\left\|\alpha\Pi_K^0 p_h^{f,n} - \psi_h^n\right\|_{0,K}^2 + c_0\left\|\Pi_K^0 p_h^{f,n}\right\|_{0,K}^2\right.$$

$$+ \tilde{\alpha}_*\left(c_0 + \frac{\alpha^2}{\lambda}\right)\left(\left\|(I - \Pi_K^0)p_h^{f,n}\right\|_{0,K}^2\right)\right) \leq \mathcal{N}.$$

Here, the term $\mathcal{N}$ is defined as,

$$\mathcal{N} := \frac{1}{2}\sum_{K\in\mathcal{T}_h}\left(\frac{1}{\lambda}\left\|\alpha\Pi_K^0 p_h^{f,0} - \psi_h^0\right\|_{0,K}^2 + c_0\left\|\Pi_K^0 p_h^{f,0}\right\|_{0,K}^2\right.$$

$$+ \tilde{\alpha}_*\left(c_0 + \frac{\alpha^2}{\lambda}\right)\left\|(I - \Pi_K^0)p_h^{f,0}\right\|_{0,K}^2\right)$$

$$+ \frac{\mu}{2}\left\|\varepsilon(\boldsymbol{u}_h^{s,0})\right\|_{0,\Omega}^2 + C\Delta t\sum_{j=1}^{n}\left(F_{\boldsymbol{b},r}^{h,j}(w_{1,h}^{j-1}, w_{2,h}^{j-1}; \delta_t\boldsymbol{u}_h^{s,j}) + G_\ell^{h,j}(p_h^{f,j})\right).$$

Use of Young's inequality gives

$$
\mathcal{N} \le \frac{\mu}{2} \left\| \boldsymbol{\varepsilon}(\boldsymbol{u}_h^{s,0}) \right\|_{0,\Omega}^2 + \frac{\kappa_1}{2\eta} \Delta t \sum_{j=1}^{n} \left\| \nabla p^{f,j} \right\|_{0,\Omega}^2 + \frac{c_0}{2} \sum_{K \in \mathcal{T}_h} \left\| \Pi_K^0 p_h^{f,0} \right\|_{0,K}^2
$$
$$
+ \frac{1}{2} \sum_{K \in \mathcal{T}_h} \left( \frac{1}{\lambda} \left\| \alpha \Pi_K^0 p_h^{f,0} - \psi_h^0 \right\|_{0,K}^2 + \tilde{\alpha}_* \left( c_0 + \frac{\alpha^2}{\lambda} \right) \left\| (I - \Pi_K^0) p_h^{f,0} \right\|_{0,K}^2 \right)
$$
$$
+ C \sum_{j=1}^{n} \left( \left\| \boldsymbol{b}^j \right\|_{0,\Omega}^2 + \Delta t \left\| \ell^j \right\|_{0,\Omega}^2 + \left\| w_{1,h}^{j-1} \right\|_{0,\Omega}^2 + \left\| w_{2,h}^{j-1} \right\|_{0,\Omega}^2 \right).
$$

Using inf-sup condition and Equation (3.2a), we obtain

$$
\tilde{\beta} \left\| \psi_h^n \right\|_{0,\Omega} \lesssim \left( \left\| \boldsymbol{b}^n \right\|_{0,\Omega} + \left\| w_{1,h}^{n-1} \right\|_{0,\Omega} + \left\| w_{2,h}^{n-1} \right\|_{0,\Omega} + \mu \left\| \boldsymbol{\varepsilon}(\boldsymbol{u}_h^{s,n}) \right\|_{0,\Omega} \right).
$$

For given $\boldsymbol{u}_h^{s,n}$ as solution of the problem (3.2a)–(3.2c), and taking $s_h = w_{1,h}^n$ in (3.3a) then the use of Young's inequality with appropriate choice of $\epsilon$ and the bound of trilinear form $c_h(\cdot; \cdot, \cdot)$ implies

$$
\frac{\Delta t}{2} \delta_t \left\| w_{1,h}^n \right\|_{0,\Omega}^2 + \Delta t\, D_1^a \left\| \nabla w_{1,h}^n \right\|_{0,\Omega}^2 \lesssim m^h(w_{1,h}^n - w_{1,h}^{n-1}, w_{1,h}^n) + \Delta t\, a_4^h(w_{1,h}^n, w_{1,h}^n)
$$
$$
\le C \Delta t (1 + \left\| w_{1,h}^{n-1} \right\|_{0,\Omega}^2 + \left\| w_{2,h}^{n-1} \right\|_{0,\Omega}^2) + \Delta t (\epsilon + C_1 \left\| \boldsymbol{u}_h^{s,n} \right\|_{1,\infty,\Omega}) \left\| w_{1,h}^n \right\|_{0,\Omega}^2.
$$

Summing over $n$ gives

$$
\left\| w_{1,h}^n \right\|_{0,\Omega}^2 + \Delta t D_1^a \sum_{j=1}^{n} \left\| \nabla w_{1,h}^j \right\|_{0,\Omega}^2 \le \left\| w_{1,h}^0 \right\|_{0,\Omega}^2 + C \left( 1 + \Delta t \sum_{j=1}^{n} \sum_{i=1}^{2} \left\| w_{i,h}^{j-1} \right\|_{0,\Omega}^2 \right)
$$
$$
+ \Delta t \sum_{j=1}^{n} (\epsilon + C_1 \left\| \boldsymbol{u}_h^{s,j} \right\|_{1,\infty,\Omega}) \left\| w_{1,h}^j \right\|_{0,\Omega}^2.
$$

Again use of these arguments for Equation (3.3b) with $s_h = w_{2,h}^n$. Then adding the resultant bounds and assuming $M := \max\limits_{1 \le j \le n} \left\| \boldsymbol{u}_h^{s,j} \right\|_{1,\infty,\Omega} < \infty$, then choosing $\epsilon, \Delta t$ so that, we have $\Delta t(\epsilon + C_1 M) < 1$. Thus, the use of Lemma 3 completes the stability of the discrete solution. $\square$

## 4   Error analysis

We will assume the following regularity assumptions to generate the convergence results: For all $t > 0$, the displacement of porous medium $\boldsymbol{u}^s(t) \in [H^2(\Omega)]^2$, the fluid pressure $p^f(t) \in H^2(\Omega)$, the total pressure $\psi(t) \in H^1(\Omega)$ and concentrations $w_1, w_2 \in H^2(\Omega)$. We will assume the regularity in time as, $\partial_t \boldsymbol{u}^s \in \mathbf{L}^2(0, T; [H^2(\Omega)]^2)$; $\partial_t p^f$ and $\partial_t \psi \in L^2(0, T; H^1(\Omega))$, $\partial_t w_1$ and $\partial_t w_2$ in $l^2(0, T; L^2(\Omega))$; $\partial_{tt} \boldsymbol{u}^s \in \mathbf{L}^2(0, T; [L^2(\Omega)]^2)$; $\partial_{tt} p^f$, $\partial_{tt} \psi$, $\partial_{tt} w_1$ and $\partial_{tt} w_2$ in $L^2(0, T; L^2(\Omega))$.

**Lemma 4.** *For each $\boldsymbol{u}^s \in \mathbf{V} \cap [H^{1+r}(\Omega)]^2$ with $0 \leq r \leq 1$ and under the regularity assumption on the polygonal mesh (mentioned in Section 3.1), there exist an interpolant $\boldsymbol{u}_I^s \in \mathbf{V}_h$ satisfying*

$$\|\boldsymbol{u}^s - \boldsymbol{u}_I^s\|_{0,\Omega} + h_K \, |\boldsymbol{u}^s - \boldsymbol{u}_I^s|_{1,\Omega} \lesssim h^{1+r}|\boldsymbol{u}^s|_{1+r,\Omega}.$$

Now, we define the following projection operator $\mathbf{A}_h := (A_h^{\boldsymbol{u}}, A_h^p, A_h^\psi, A_h^{w_1}, A_h^{w_2})$, where $(A_h^{\boldsymbol{u}}, A_h^\psi)$ and $A_h^p, A_h^{w_1}, A_h^{w_2}$ are standard Stokes and elliptic projection operators respectively.

For $\mathbb{V} := \mathbf{V} \times Q \times Z \times W \times W$ and $\mathbb{V}_h := \mathbf{V}_h \times Q_h \times Z_h \times W_h \times W_h$, these operators satisfy the following estimates (see, for instance, [27, 29]):

**Lemma 5.** *Let $(\boldsymbol{u}^s, p^f, \psi, w_1, w_2) \in \mathbb{V}$, $(A_h^{\boldsymbol{u}}\boldsymbol{u}^s, A_h^p p^f, A_h^\psi \psi, A_h^{w_1} w_1, A_h^{w_2} w_2) \in \mathbb{V}_h$ be the unique solution to the system of equations (2.1) and projection operators, respectively. Then,*

$$\|\boldsymbol{u}^s - A_h^{\boldsymbol{u}}\boldsymbol{u}^s\|_{0,\Omega} + h\Big(|\boldsymbol{u}^s - A_h^{\boldsymbol{u}}\boldsymbol{u}^s|_{1,\Omega} + \big\|\psi - A_h^\psi \psi\big\|_{0,\Omega}\Big)$$

$$\lesssim h^2(|\boldsymbol{u}|_{2,\Omega} + |\psi|_{1,\Omega}), \tag{4.1a}$$

$$\big\|p^f - A_h^p p^f\big\|_{0,\Omega} + h|p^f - A_h^p p^f|_{1,\Omega} \lesssim h^2 |p|_{2,\Omega},$$

$$\|w_i - A_h^{w_i} w_i\|_{0,\Omega} + h|w_i - A_h^{w_i} w_i|_{1,\Omega} \lesssim h^2 |w_i|_{2,\Omega}, \quad i = 1, 2.$$

To derive the theoretical error estimates for fully discrete scheme, we decompose the error using the projection operator $\mathbf{A}_h$ as follows :

$$\xi(t_n) - \xi_h^n = (\xi(t_n) - \mathbf{A}_h \xi(t_n)) + (\mathbf{A}_h \xi(t_n) - \xi_h^n) := \rho_\xi^n + \eta_\xi^n,$$

for $\xi(t_n) = \boldsymbol{u}^s(t_n), p^f(t_n), \psi(t_n), w_1(t_n), w_2(t_n)$, $\xi_h^n = \boldsymbol{u}_h^{s,n}, p_h^{f,n}, \psi_h^n, w_{1,h}^n, w_{2,h}^n$ and for each $n = 1, \ldots, N$.

From the weak formulation (2.1) and fully discrete scheme (3.2)–(3.3), and an appeal to projection operator $\mathbf{A}_h$, the error equations for the fully discrete scheme in terms of $\eta_\xi^n$, where $\xi = \boldsymbol{u}, p, \psi, w_1, w_2$, are

$$a_1^h(\eta_{\boldsymbol{u}}^n, \boldsymbol{v}_h^s) + b_1(\boldsymbol{v}_h^s, \eta_\psi^n) = F_{\boldsymbol{b},r}^n(w_1^n, w_2^n; \boldsymbol{v}_h^s) - F_{\boldsymbol{b},r}^{h,n}(w_{1,h}^{n-1}, w_{2,h}^{n-1}; \boldsymbol{v}_h^s), \tag{4.2a}$$

$$\tilde{a}_2^h(\delta_t \eta_p^n, q_h^f) + a_2^h(\eta_p^n, q_h^f) - b_2(q_h^f, \delta_t \eta_\psi^n) = (G_\ell^n - G_\ell^{h,n})(q_h^f)$$

$$+ b_2(q_h^f, \delta_t A_h^\psi \psi^n - \partial_t \psi^n) + \Big(\tilde{a}_2^h(\delta_t A_h^p p^n, q_h^f) - \tilde{a}_2(\partial_t p^n, q_h^f)\Big), \tag{4.2b}$$

$$b_1(\delta_t \eta_{\boldsymbol{u}}^n, \phi_h) + b_2(\delta_t \eta_p^n, \phi_h) - a_3(\delta_t \eta_\psi^n, \phi_h) = -b_2(\delta_t \rho_p^n, \phi_h) + a_3(\delta_t \rho_\psi^n, \phi_h), \tag{4.2c}$$

$$m^h(\delta_t \eta_{w_i}^n, s_h) + a_{3+i}^h(\eta_{w_i}^n, s_h) = \big(m^h(\delta_t A_h^{w_i} w_i^n, s_h) - m(\partial_t w_i^n, s_h)\big)$$

$$+ \Big(\big(J_z^n(w_1^n, w_2^n, \boldsymbol{u}^{s,n}; s_h) - J_z^{h,n}(w_{1,h}^{n-1}, w_{2,h}^{n-1}, \boldsymbol{u}_h^{s,n}; s_h)\big)$$

$$- \big(c(\boldsymbol{u}^{s,j}; w_i^j, s_h) - c^h(\boldsymbol{u}_h^{s,j}; w_{i,h}^j, s_h)\big)\Big), \tag{4.2d}$$

for $i = 1, z = f$ and $i = 2, z = g$.

Now, we will divide the derivation of the error estimates of the fully discrete scheme (3.2)–(3.3) into the two lemmas: one containing the error bounds from

the poroelasticity equations and other-regarding ADR equations. We start here by recalling/mentioning the bounds to be used in succeeding lemmas.

The use of Taylor's expansion shows the following corollary holds,

*Corollary 1.* For any $\xi$, we have

$$\sum_{j=1}^{n} \left\| \xi^j - \xi^{j-1} \right\|_{0,\Omega}^2 \lesssim (\Delta t)^2 \Big( \sum_{j=1}^{n} \left\| \partial_t \xi^j \right\|_{0,\Omega}^2 + \left\| \partial_{tt} \xi(s) \right\|_{L^1(L^2(\Omega))}^2 \Big), \quad (4.3a)$$

$$\Delta t \sum_{j=1}^{n} \left\| \partial_t \xi^j - \delta_t \xi^j \right\|_{0,\Omega}^2 \lesssim (\Delta t)^3 \left\| \partial_{tt} \xi \right\|_{L^2(L^2(\Omega))}^2. \quad (4.3b)$$

Now, we derive the result below by using the Lipshitz condition of function $r(w_1^n, w_2^n)$ for each $n$, Corollary 1 and standard inequalities.

**Lemma 6 [Coupled poroelastic error bounds].** *Let* $(\boldsymbol{u}^{s,n}, p^{f,n}, \psi^n) \in \mathbf{V} \times Q \times Z$ *be the solution to the system* (2.1) *and* $(\boldsymbol{u}_h^{s,n}, p_h^{f,n}, \psi_h^n) \in \mathbf{V}_h \times Q_h \times Z_h$ *be the solution to the system* (3.2) *for each* $n = 1, \dots, N$. *Then the following estimate holds, with constant $C$ independent of $h$ and $\Delta t$,*

$$\| \varepsilon(\eta_{\boldsymbol{u}}^n) \|_{0,\Omega}^2 + \| \eta_\psi^n \|_{0,\Omega}^2 + \| \eta_p \|_{l^2(H^1(\Omega))}^2$$

$$\leq C \Big( h^2 + \Delta t^2 + \sum_{j=1}^{n} \sum_{i=1}^{2} \left\| \eta_{w_i}^{j-1} \right\|_{0,\Omega}^2 \Big) + 2\epsilon_3 \Delta t \sum_{j=1}^{n} \left\| \eta_\psi^j \right\|_{0,\Omega}^2, \quad (4.4)$$

*where $\epsilon_3 > 0$ chosen in subsequent analysis.*

*Proof.* Set $\boldsymbol{v}_h^s = \delta_t \eta_{\boldsymbol{u}}^n$ in (4.2a), $q_h^f = \eta_p^n$ in (4.2b), and $\phi_h = \eta_\psi^n$ in (4.2c) then summing over each $n$ the resultant equation with the use of Lemma 2 gives

$$\Delta t \sum_{j=1}^{n} \Big( \delta_t a_1^h(\eta_{\boldsymbol{u}}^j, \eta_{\boldsymbol{u}}^j) + \Delta t \ a_1^h(\delta_t \eta_{\boldsymbol{u}}^j, \delta_t \eta_{\boldsymbol{u}}^j) + a_2^h(\eta_p^j, \eta_p^j) + L_h^j \Big)$$

$$= \Delta t \sum_{j=1}^{n} \Big( \big( F_{\boldsymbol{b},r}^j(w_1^j, w_2^j; \delta_t \eta_{\boldsymbol{u}}^j) - F_{\boldsymbol{b},r}^{h,j}(w_{1,h}^{j-1}, w_{2,h}^{j-1}; \delta_t \eta_{\boldsymbol{u}}^j) \big) + \big( (G_\ell^j - G_\ell^{h,j})(\eta_p^j)$$

$$+ \big( \tilde{a}_2^h(\delta_t A_h^p p^j, \eta_p^j) - \tilde{a}_2(\partial_t p^j, \eta_p^j) \big) \big) + b_2(\eta_p^j, \delta_t A_h^\psi \psi^j - \partial_t \psi^j)$$

$$+ \big( b_2(\delta_t \rho_p^j, \eta_\psi^j) - a_3(\delta_t \rho_\psi^j, \eta_\psi^j) \big) \Big) := \sum_{i=1}^{4} R_i.$$

The error term $R_1$ can be written as

$$R_1 = \Delta t \sum_{j=1}^{n} \sum_{K \in \mathcal{T}_h} \Big( \rho \Big( (I - \boldsymbol{\Pi}_K^{0,0}) \boldsymbol{b}^j, \delta_t \eta_{\boldsymbol{u}}^j \Big)_{0,K}$$

$$+ \tau \Big\langle (r(w_1^j, w_2^j) - r_h(w_{1,h}^{j-1}, w_{2,h}^{j-1}))(\boldsymbol{k} \otimes \boldsymbol{k}), \delta_t \varepsilon(\eta_{\boldsymbol{u}}^j) \Big\rangle_{0,K} \Big) := R_1^a + R_1^b.$$

The Young's inequality with some constant $\epsilon_1 > 0$ gives

$$R_1^a \le Ch^2 \sum_{j=1}^n |\boldsymbol{b}^j|_{1,\Omega}^2 + \epsilon_1 \Delta t^2 \sum_{j=1}^n \left\|\boldsymbol{\varepsilon}(\delta_t \eta_{\boldsymbol{u}}^j)\right\|_{0,\Omega}^2.$$

The bound of $R_1^b$ is through the property of function $r(w_1, w_2)$, triangle's inequality, and bound (4.3a) yields

$$R_1^b \le \epsilon_1 \Delta t^2 \sum_{j=1}^n \left\|\boldsymbol{\varepsilon}(\delta_t \eta_{\boldsymbol{u}}^j)\right\|_{0,\Omega}^2 + C\tau \sum_{j=1}^n \left\|(r(w_1^j, w_2^j) - r_h(w_{1,h}^{j-1}, w_{2,h}^{j-1}))\right\|_{0,\Omega}^2,$$

$$\le \epsilon_1 \Delta t^2 \sum_{j=0}^n \left\|\boldsymbol{\varepsilon}(\delta_t \eta_{\boldsymbol{u}}^j)\right\|_{0,\Omega}^2 + C\tau^2 \sum_{j=1}^n \sum_{i=1}^2 \left\|\eta_{w_i}^{j-1}\right\|_{0,\Omega}^2 + C\tau^2 h^2 \sum_{i=1}^2 \sum_{j=0}^{n-1} |w_i^j|_{1,\Omega}^2$$

$$+ C\tau^2 (\Delta t)^2 \sum_{i=1}^2 \left( \sum_{j=1}^n \left\|\partial_t w_i^j\right\|_{0,\Omega}^2 + \|\partial_{tt} w_i(s)\|_{L^1(L^2(\Omega))}^2 \right).$$

Use of polynomial approximation and consistency of $\tilde{a}_2^{h,K}(\cdot,\cdot)$, using Cauchy Schwarz, Poincaré and Young's inequalities, estimate (4.3b) and by Taylor's expansion, we get

$$R_2 \le C\left(h^2\left(\left\|\partial_t p^f\right\|_{L^2(H^1(\Omega))}^2 + \|\ell\|_{l^2(H^1(\Omega))}^2\right) + (\Delta t)^3 \left\|\partial_{tt} p^f\right\|_{L^2(L^2(\Omega))}^2\right)$$

$$+ \epsilon_2 \Delta t \sum_{j=1}^n \left\|\nabla \eta_p^j\right\|_{0,\Omega}^2.$$

An application of Lemma 1 and Young's inequality implies

$$R_3 \le Ch^2 \int_0^{t_n} \|\partial_t \psi(s)\|_{1,\Omega}^2 \, \mathrm{d}s + C(\Delta t)^3 \|\partial_{tt} \psi\|_{L^2(L^2(\Omega))}^2 + \epsilon_2 \Delta t \sum_{j=1}^n \left\|\nabla \eta_p^j\right\|_{0,\Omega}^2.$$

Use of the estimates (4.1a) and Lemma 1 as seen in bound of $R_2$ with constant $\epsilon_3 > 0$ gives

$$R_4 \le C \, h^2 \left(\left\|\partial_t p^f\right\|_{L^2(H^1(\Omega))}^2 + \|\partial_t \psi\|_{L^2(H^1(\Omega))}^2\right) + \epsilon_3 \Delta t \sum_{j=1}^n \left\|\eta_\psi^j\right\|_{0,\Omega}^2.$$

Combining the bounds of $R_i$'s, we get

$$\sum_{i=1}^4 R_i \le C(h^2 + \Delta t^2) + 2\epsilon_1 \left\|\boldsymbol{\varepsilon}(\eta_{\boldsymbol{u}}^n)\right\|_{0,\Omega}^2 + 5\epsilon_2 \Delta t \sum_{j=1}^n \left\|\nabla \eta_p^j\right\|_{0,\Omega}^2$$

$$+ 2\epsilon_3 \Delta t \sum_{j=1}^n \left\|\eta_\psi^j\right\|_{0,\Omega}^2 + C \sum_{j=1}^n \sum_{i=1}^2 \left\|\eta_{w_i}^{j-1}\right\|_{0,\Omega}^2.$$

Using inf-sup condition of $b_1(\cdot,\cdot)$ and error equation (4.2a), we obtain

$$\left\|\eta_\psi^n\right\|_{0,\Omega} \le C\Big(h(|\boldsymbol{b}^n|_{1,\Omega} + \sum_{i=1}^2 |w_i^{n-1}|_{1,\Omega}) + (\Delta t)\Big(\mu\left\|\boldsymbol{\varepsilon}(\eta_{\boldsymbol{u}}^n)\right\|_{0,\Omega}$$

$$+ \sum_{i=1}^2 \big(\left\|\partial_t w_i^n\right\|_{0,\Omega} + \left\|\partial_{tt} w_i(s)\right\|_{L^1(t_{n-1},t_n;L^2(\Omega))} + \left\|\eta_{w_i}^{n-1}\right\|_{0,K}\big)\Big)\Big)\left\|\boldsymbol{v}_h^s\right\|_{1,\Omega}.$$

Therefore, the adequate choices of $\epsilon$'s and choosing the initial conditions using projections as

$$\boldsymbol{u}_h^{s,0} := \boldsymbol{u}_I^s(0) \text{ and } p_h^{f,0} := p_I^f(0)$$

conclude the error bounds (4.4). $\square$

Note that the initial conditions are chosen so that the estimates of $\eta_{\boldsymbol{u}}^n, \eta_p^n$ and $\eta_\psi^n$ are known, and can consider another such choices for analysis and computations. Next, we approach the remaining error equations to avail the following lemma.

**Lemma 7 [Coupled CDR error bounds].** *Let $(w_1^n, w_2^n) \in [W]^2$ be the solution to the continuous problem (2.2)–(2.3) and $(w_{1,h}^n, w_{2,h}^n) \in [W_h]^2$ be the solution to the discrete problem (3.3) for each $n$. Then the following estimate holds, with constant $C$ independent of $h$ and $\Delta t$,*

$$\sum_{i=1}^2 \Big(\left\|\eta_{w_i}^n\right\|_{0,\Omega}^2 + D_1^a(\Delta t)\sum_{j=1}^n |\eta_{w_i}^j|_{1,\Omega}^2\Big) \le C(h^2 + \Delta t^2 + \sum_{j=1}^n \left\|\eta_{\boldsymbol{u}}^{j-1}\right\|_{0,\Omega}^2$$

$$+ \sum_{j=1}^n \sum_{i=1}^2 \left\|\eta_{w_i}^{j-1}\right\|_{0,\Omega}^2) + \Delta t^2 \sum_{j=1}^n (2\epsilon_1\left\|\boldsymbol{\varepsilon}(\eta_{\boldsymbol{u}}^j)\right\|_{0,\Omega}^2 + 6\epsilon_4|\eta_{w_1}^j|_{1,\Omega}^2).$$

*Proof.* Taking $s_h = \eta_{w_1}^n$ in (4.2d) for $i = 1$, then multiplying with $\Delta t$ and summing over $n$ enable us to get

$$(\left\|\eta_{w_1}^n\right\|_{0,\Omega}^2 - \left\|\eta_{w_1}^0\right\|_{0,\Omega}^2) + D_1^a \Delta t \sum_{j=1}^n \left\|\nabla\eta_{w_1}^j\right\|_{0,\Omega}^2$$

$$\lesssim \Delta t \sum_{j=1}^n \Big(\big(J_f^j(w_1^j, w_2^j, \boldsymbol{u}^{s,j}; \eta_{w_1}^j) - J_f^{h,j}(w_{1,h}^{j-1}, w_{2,h}^{j-1}, \boldsymbol{u}_h^{s,j}; \eta_{w_1}^j)\big)$$

$$- \big(m(\partial_t w_1^j, \eta_{w_1}^j) - m^h(\delta_t A_h^{w_1} w_1^j, \eta_{w_1}^j)\big)$$

$$- \big(c(\boldsymbol{u}^{s,j}; w_1^j, \eta_{w_1}^j) - c^h(\boldsymbol{u}_h^{s,j}; w_{1,h}^j, \eta_{w_1}^j)\big)\Big) := A_1 + A_2 + A_3.$$

Use of estimates for projection $\Pi_K^0$, triangle's inequality and projection (4.1) gives

$$A_1 \lesssim \Delta t \sum_{j=1}^n \Big(\left\|\rho_{\boldsymbol{u}}^j\right\|_{0,\Omega} + \left\|\eta_{\boldsymbol{u}}^j\right\|_{0,\Omega} + h|f(w_1^{j-1}, w_2^{j-1}, \boldsymbol{u}^{s,j})|_{1,\Omega}$$

$$+ \sum_{i=1}^2 \big(\left\|\rho_{w_i}^{j-1}\right\|_{0,\Omega} + \left\|\eta_{w_i}^{j-1}\right\|_{0,\Omega} + \left\|w_i^j - w_i^{j-1}\right\|_{0,\Omega}\big)\Big)\left\|\eta_{w_1}^j\right\|_{0,\Omega}.$$

The estimate (4.3a), and Young's inequality with $\epsilon_1\epsilon_4 = C^2/4$ and $\epsilon_1, \epsilon_4 > 0$ gives

$$A_1 \leq C \sum_{j=1}^{n}\sum_{i=1}^{2} \left\|\eta_{w_i}^{j-1}\right\|_{0,\Omega}^2 + \epsilon_1\,\Delta t\left\|\eta_{\boldsymbol{u}}^{j}\right\|_{0,\Omega}^2 + \epsilon_4\,\Delta t\sum_{j=1}^{n}\left\|\eta_{w_1}^{j}\right\|_{0,\Omega}^2$$

$$+ C\sum_{j=1}^{n}\left(h^2\left(|\boldsymbol{u}^{s,j}|_{2,\Omega}^2 + |f(w_1^{j-1}, w_2^{j-1}, \boldsymbol{u}^{s,j})|_{1,\Omega}^2 + \sum_{i=1}^{2}|w_i^{j-1}|_{2,\Omega}^2\right)\right.$$

$$\left.+ (\Delta t)^2\sum_{i=1}^{2}\left(\left\|\partial_t w_i^j\right\|_{0,\Omega} + \int_{t_{j-1}}^{t_j}\|\partial_{tt}w_i(s)\|_{0,\Omega}\,\mathrm{d}s\right)^2\right).$$

The use of consistency for bilinear form $m^h(\cdot,\cdot)$ and bounds (4.3a)–(4.3b) with Young's inequality for $\epsilon_4 > 0$ gives

$$A_2 \leq C\left(h^2\int_0^{t_n}|\partial_t w_1(s)|_{1,\Omega}^2\,\mathrm{d}s + (\Delta t)^2\int_0^{t_n}\|\partial_{tt}w_1(s)\|_{0,\Omega}^2\,\mathrm{d}s\right)$$

$$+ \epsilon_4\Delta t\sum_{j=1}^{n}\left\|\nabla\eta_{w_1}^{j}\right\|_{0,\Omega}^2.$$

Assume that $\nabla w_1^j$ and $\boldsymbol{u}_h^{s,j}$ are bounded for each $j$ then the values of $\left\|\nabla w_1^j\right\|_{\infty,K}$ and $\left\|\Pi_K^0\boldsymbol{u}_h^{s,j}\right\|_{\infty,K}$ (by use of inverse estimate) are finite respectively.

$$A_3 \lesssim \Delta t\sum_{j=1}^{n}\sum_{K\in\mathcal{T}_h}\left(\left\|(I-\Pi_K^0)(\boldsymbol{u}^{s,j}\cdot\nabla w_1^j)\right\|_{0,K} + \left\|(I-\Pi_K^0)\boldsymbol{u}^{s,j}\right\|_{0,K}\right.$$

$$\left.+ \left\|(I-\boldsymbol{\Pi}_K^{0,0})\nabla w_1^j\right\|_{0,K} + \|\rho_{\boldsymbol{u}}^j\|_{0,K} + \|\eta_{\boldsymbol{u}}^j\|_{0,K} + |\rho_{w_1}^j|_{1,K} + |\eta_{w_1}^j|_{1,K}\right)\left\|\eta_{w_1}^j\right\|_{0,K}.$$

Thus, by applying the Cauchy-Schwarz and Young's inequalities implies

$$A_3 \leq C\,h^2\sum_{j=1}^{n}\left(\left\|(\boldsymbol{u}^{s,j}\cdot\nabla w_1^j)\right\|_{1,\Omega}^2 + |\boldsymbol{u}^{s,j}|_{1,\Omega}^2 + |\nabla w_1^j|_{1,\Omega}^2\right)$$

$$+ \Delta t^2\sum_{j=1}^{n}(\epsilon_1\left\|\eta_{\boldsymbol{u}}^j\right\|_{0,\Omega}^2 + \epsilon_4|\eta_{w_1}^j|_{1,\Omega}^2).$$

Thus, the bounds of $A_i$'s gives

$$\|\eta_{w_1}^n\|_{0,\Omega}^2 + \Delta t\sum_{j=1}^{n}\left\|\nabla\eta_{w_1}^j\right\|_{0,\Omega}^2 \leq C\left(h^2 + \Delta t^2 + \left\|\eta_{w_1}^0\right\|_{0,\Omega}^2 + \sum_{j=1}^{n}\sum_{i=1}^{2}\left\|\eta_{w_i}^{j-1}\right\|_{0,\Omega}^2\right)$$

$$+ \Delta t\sum_{j=1}^{n}(\epsilon_1\left\|\eta_{\boldsymbol{u}}^j\right\|_{0,\Omega}^2 + 3\epsilon_4|\eta_{w_1}^j|_{1,\Omega}^2).$$

Similar to the above bounds, taking $s_h = \eta_{w_2}^n$ in (4.2d) for $i = 2$, we get the error bounds in the terms of $\eta_{w_2}$. Thus, the addition of the bounds for $w_1$ and $w_2$ concludes the proof. $\square$

Application of the discrete Gronwall's inequality (Lemma 3.5) with the combination of results from Lemmas 6–7 yield the final result:

**Theorem 3 [Fully-discrete error estimates].** *Let $\boldsymbol{u}^s(t_n) \in \mathbf{V}$, $p^f(t_n) \in Q$, $\psi(t_n) \in Z$, $w_1(t_n), w_2(t_n) \in W$ be the solution to continuous problem (2.1), and $\boldsymbol{u}_h^{s,n} \in \mathbf{V}_h$, $p_h^{f,n} \in Q_h$, $\psi_h^n \in Z_h$, $w_{1,h}^n, w_{2,h}^n \in W_h$ be the solution to fully discrete problem (3.2)–(3.3) for each $n = 1, \ldots, N$. Then the following estimate holds, with constant $C$ independent of $h$ and $\Delta t$,*

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}^s(t_n) - \boldsymbol{u}_h^{s,n})\|_{0,\Omega}^2 + \|\psi(t_n) - \psi_h^n\|_{0,\Omega}^2 + \|p^f - p_h^f\|_{l^2(H^1(\Omega))}^2$$

$$+ \sum_{i=1}^{2} \|w_i - w_{i,h}\|_{l^2(H^1(\Omega))}^2 \lesssim (\Delta t^2 + h^2).$$

# 5 Numerical experiments

We define the $L^2$ and $H^1$ errors for the approximation space of order $k$ as

$$E_0(v) := \sum_{K \in \mathcal{T}_h} \left\| v - \Pi_K^0 v_h \right\|_{0,K}, E_1(v) := \sum_{K \in \mathcal{T}_h} \left\| \nabla v - \boldsymbol{\Pi}_K^{0,0} \nabla v_h \right\|_{0,K},$$

where $v$ can be $\boldsymbol{u}^s$, $p^f$, $\psi$ and $w_1, w_2$. The convergence rates of the errors $E_k(v)$ and $E_k'(v)$ with $k = 0, 1$ for the corresponding mesh sizes $h$ and $h'$ respectively, are calculated as $r_k(v) = \log(E_k(v)/E_k'(v)) / \log(h/h')$.

## 5.1 Space and time convergence

We initiate the tests in the domain $\Omega_1 := (0,1) \times (0,1)$ and verify the spatial convergence rate of the VEM for given exact solutions by discretizing the domain into elements containing non-convex polygons seen in Figures 1–3.
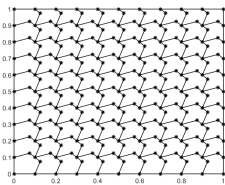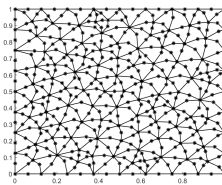


**Figure 1.** Concave mesh $\mathcal{N}_h$.

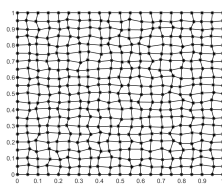**Figure 2.** Distorted triangular mesh $\mathcal{H}_h$.

**Figure 3.** Distorted square mesh $\mathcal{D}_h$.

For this, we consider the following exact solutions for global displacement and fluid pressure,

$$\boldsymbol{u}^s(x,y,t) = \begin{pmatrix} t\left(-\cos(2\pi x)\sin(2\pi y) + \sin(2\pi y) + \sin^2(\pi x)\sin^2(\pi y)\right) \\ t\left(\sin(2\pi x)\cos(2\pi y) - \sin(2\pi x)\right) \end{pmatrix},$$

$$p^f(x,y,t) = t\sin^2(\pi x)\sin^2(\pi y),$$

together with the parametric values $\nu = 0.3$, $E = 100$, $\kappa = 1$, $\alpha = 1$, $c_0 = 1$, $\eta = 0.1$, $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$, $\mu = \frac{E}{2(1+\nu)}$, and scalar function $r(w_1, w_2) := w_1 + w_2$. Also, the exact concentration solutions are given as $w_1(x, y, t) = w_2(x, y, t) = t\sin(\pi x)\sin(\pi y)$, and the reaction kinematics with unit value of $D_1, D_2, \beta_1, \beta_2, \beta_3$ and $\gamma = 0.1$ making the concentration equations with different load functions. However, the load functions $\ell$ and $\boldsymbol{b}$, and the exact global pressure $\psi$ is obtained from the problem (1.1) in domain $\Omega$ with $\tau = 1, \rho = 1$. We see that the trend of the computed errors with $h = \Delta t$ in the Figure 4 is the one expected from the theoretical results in Section 4.
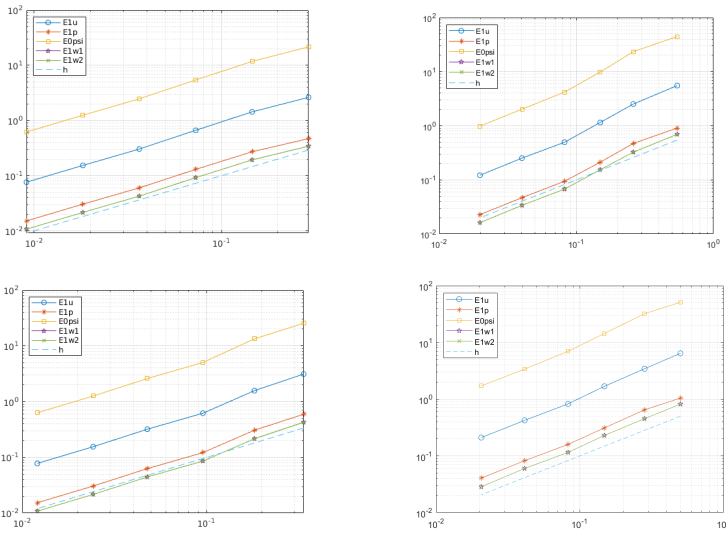


**Figure 4.** Computed errors for meshes $\mathcal{N}_h$, $\mathcal{H}_h$ and $\mathcal{D}_h$ in first three figures, and last one for variable $\kappa = x + y + 10$ on mesh $\mathcal{H}_h$.

## 5.2   Space convergence with mixed boundary conditions

For this, we consider the following exact solutions for global displacement and fluid pressure

$$\boldsymbol{u}^s(x, y, t) = \begin{pmatrix} \exp(-t)\sin(\pi x)\sin(\pi y) \\ \exp(-t)\sin(\pi x)\sin(\pi y) \end{pmatrix},$$

$$p^f(x, y, t) = \exp(-t)\sin(\pi x)(1 + \cos(\pi y)),$$

together with the parametric values $\nu = 0.495$, $E = 100$, $\kappa = 0.5$, $\alpha = 0.1$, $c_0 = 1e - 03$, $\eta = 0.1$ and scalar function $r(w_1, w_2) := w_1 + w_2$. Also, the exact concentration solutions are given as

$$w_1(x, y, t) = \exp(-t)(-\cos(2\pi x)\sin(2\pi y) + \sin(2\pi y) + \sin^2(\pi x)\sin^2(\pi y)),$$

$$w_2(x, y, t) = \exp(-t)(\sin(2\pi x)\cos(2\pi y) - \sin(2\pi x)),$$

taking the diffusion constants $D_1 = 0.01$, $D_2 = 1$, and the reaction kinematics with $\gamma = 0.0001$, $\beta_3 = 0.80$, $\beta_1$, $\beta_2 = 0.15$. However the load functions $\ell, \boldsymbol{b}$ and the exact global pressure $\psi$ is obtained from the respective problem (1.1) in domain $\Omega$ with $\tau = 10, \rho = 1$. We can note that we have considered the parametric values to check the extend of the method with small $c_0$ and $D_1$, and large values of $\lambda$. We show the computed rate of convergence with $\Delta t = 0.005$ and varying mesh sizes $h$ on a distorted triangular meshes (shown in Figure 2) in the Table 1. We have computed the relative errors $\bar{E}_k$ for any variable $v$ as $\bar{E}_k(v) = E_k(v)/\left\|v\right\|_{k,\Omega}$ and $r_k(v)$ as their rate of convergence.

**Table 1.** Computed relative errors and their rate of convergence with mesh size $h$.

| $h^{-1}$ | $\bar{E}_1(\boldsymbol{u})$ | $r_1(\boldsymbol{u})$ | $\bar{E}_1(p)$ | $r_1(p)$ | $\bar{E}_0(\psi)$ | $r_0(\psi)$ | $\bar{E}_1(w_1)$ | $r_1(\boldsymbol{w}_1)$ | $\bar{E}_1(w_2)$ | $r_1(\boldsymbol{w}_2)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0.9720 | – | 0.5232 | – | 0.9774 | – | 0.4283 | – | 0.3479 | – |
| 20 | 0.5514 | 0.82 | 0.3374 | 0.63 | 0.3909 | 1.32 | 0.2352 | 0.86 | 0.2040 | 0.77 |
| 40 | 0.2913 | 0.92 | 0.1614 | 1.06 | 0.1649 | 1.24 | 0.1074 | 1.13 | 0.1008 | 1.01 |
| 80 | 0.1490 | 0.97 | 0.0817 | 0.98 | 0.0814 | 1.02 | 0.0516 | 1.06 | 0.0506 | 0.99 |
| 160 | 0.0751 | 0.99 | 0.0407 | 1.01 | 0.0404 | 1.01 | 0.0255 | 1.02 | 0.0254 | 1.00 |
| 320 | 0.0380 | 0.98 | 0.0202 | 1.01 | 0.0200 | 1.01 | 0.0128 | 0.99 | 0.0123 | 1.04 |

## 6    Conclusions

By extending the analysis of [12,29], we have discussed and analyzed the lowest order conforming virtual element methods for the approximation of coupled poroelasticity and ADR equations. The major contributions of this article are: showing the well-posedness of fully discrete schemes and establishing the optimal *a-priori* error estimates for all the variables that naturally appeared in the weak formulation. The possible extensions of this work include the study of general flow-transport problems, and the coupling with other phenomena such as diffusion of solutes in poroelastic structures [28], interface elasticity-poroelasticity problems [3,16], multilayer poromechanics, or multiple-network consolidation models [21].

## References

[1]  B. Ahmad, A. Alsaedi, F. Brezzi, L.D. Marini and A. Russo. Equivalent projectors for virtual element methods. *Computers & Mathematics with Applications*, **66**(3):376–391, 2013. https://doi.org/10.1016/j.camwa.2013.05.015.

[2] V. Anaya, M. Bendahmane, D. Mora and R. Ruiz Baier. On a vorticity-based formulation for reaction-diffusion-Brinkman systems. *Networks and Heterogeneous Media*, **13**(1):69–94, 2018. https://doi.org/10.3934/nhm.2018004.

[3] V. Anaya, Z. de Wijn, B. Gómez-Vargas, D. Mora and R. Ruiz-Baier. Rotation-based mixed formulations for an elasticity-poroelasticity interface problem. *SIAM Journal on Scientific Computing*, **42**(1):B225–B249, 2020. https://doi.org/10.1137/19M1268343.

[4] P.F. Antonietti, L. Beirão da Veiga, D. Mora and M. Verani. A stream virtual element formulation of the Stokes problem on polygonal meshes. *SIAM Journal on Numerical Analysis*, **52**(1):386–404, 2014. https://doi.org/10.1137/13091141X.

[5] Beir ao da Veiga L., Brezzi F., A. Cangiani, G. Manzini, L.D. Marini and A. Russo. Basic principles of virtual element methods. *Mathematical Models and Methods in Applied Sciences*, **23**(1):199–214, 2013. https://doi.org/10.1142/S0218202512500492.

[6] L. Beirão da Veiga, F. Brezzi, L.D. Marini and A. Russo. Virtual element implementation for general elliptic equations. In *Building bridges: connections and challenges in modern approaches to numerical partial differential equations*, volume 114 of *Lecture Notes in Computational Science and Engineering*, pp. 39–71. Springer, 2016. https://doi.org/10.1007/978-3-319-41640-3_2.

[7] L. Beirão da Veiga, C. Lovadina and G. Vacca. Divergence free virtual elements for the Stokes problem on polygonal meshes. *ESAIM. Mathematical Modelling and Numerical Analysis*, **51**(2):509–535, 2017. https://doi.org/10.1051/m2an/2016032.

[8] L. Beirão da Veiga, C. Lovadina and G. Vacca. Virtual elements for the Navier–Stokes problem on polygonal meshes. *SIAM Journal on Numerical Analysis*, **56**(3):1210–1242, 2018. https://doi.org/10.1137/17M1132811.

[9] L. Beirão da Veiga, A. Pichler and G. Vacca. A virtual element method for the miscible displacement of incompressible fluids in porous media. *Computer Methods in Applied Mechanics and Engineering*, **375**:113649, 2021. https://doi.org/10.1016/j.cma.2020.113649.

[10] D. Boffi, M. Botti and D. Di Pietro. A nonconforming high-order method for the Biot problem on general meshes. *SIAM Journal on Scientific Computing*, **38**(3):A1508–A1537, 2016.

[11] M. Botti, D. Di Pietro and P. Sochala. A hybrid high-order discretization method for nonlinear poroelasticity. *Computational Methods in Applied Mathematics*, **20**(2):227–249, 2020. https://doi.org/10.1515/cmam-2018-0142.

[12] R. Bürger, S. Kumar, D. Mora, R. Ruiz-Baier and N. Verma. Virtual element methods for the three-field formulation of time-dependent linear poroelasticity. *Advances in Computational Mathematics volume*, **47**(2), 2021. https://doi.org/10.1007/s10444-020-09826-7.

[13] A. Cangiani, P. Chatzipantelidis, G. Diwan and E.H. Georgoulis. Virtual element method for quasilinear elliptic problems. *IMA Journal of Numerical Analysis*, **40**(4):2450–2472, 2020. https://doi.org/10.1093/imanum/drz035.

[14] J. Coulet, I. Faille, V. Girault, N. Guy and F. Nataf. A fully coupled scheme using virtual element method and finite volume for poroelasticity. *Computational Geosciences*, **24**(2):381–403, 2020. https://doi.org/10.1007/s10596-019-09831-w.

[15] L.M. De Oliveira Vilaca, B. Gómez-Vargas, S. Kumar, R. Ruiz-Baier and N. Verma. Stability analysis for a new model of multi-species convection-diffusion-reaction in poroelastic tissue. *Applied Mathematical Modelling. Simulation and Computation for Engineering and Environmental Systems*, **84**:425–446, 2020. https://doi.org/10.1016/j.apm.2020.04.014.

[16] V. Girault, G. Pencheva, M.F. Wheeler and T. Wildey. Domain decomposition for poroelasticity and elasticity with DG jumps and mortars. *Mathematical Models and Methods in Applied Sciences*, **21**(1):169–213, 2011. https://doi.org/10.1142/S0218202511005039.

[17] V. Girault and P.-A. Raviart. *Finite element approximation of the Navier-Stokes equations.* Lecture Notes in Mathematics. Springer-Verlag, 1979. https://doi.org/10.1007/BFb0063447.

[18] J. Heywood and R. Rannacher. Finite-element approximation of the nonstationary Navier-Stokes problem. IV. Error analysis for second-order time discretization. *SIAM Journal on Numerical Analysis*, **27**(2):353–384, 1990. https://doi.org/10.1137/0727022.

[19] S. Kumar, R. Oyarzúa, R. Ruiz-Baier and R. Sandilya. Conservative discontinuous finite volume and mixed schemes for a new four-field formulation in poroelasticity. *ESAIM. Mathematical Modelling and Numerical Analysis*, **54**(1):273–299, 2020. https://doi.org/10.1051/m2an/2019063.

[20] J. Lee, K. Mardal and R. Winther. Parameter-robust discretization and preconditioning of Biot's consolidation model. *SIAM Journal on Scientific Computing*, **39**(1):A1–A24, 2017. https://doi.org/10.1137/15M1029473.

[21] J. Lee, E. Piersanti, K.-A. Mardal and M.E. Rognes. A mixed finite element method for nearly incompressible multiple-network poroelasticity. *SIAM Journal on Scientific Computing*, **41**(2):A722–A747, 2019. https://doi.org/10.1137/18M1182395.

[22] R. Oyarzúa and R. Ruiz-Baier. Locking-free finite element methods for poroelasticity. *SIAM Journal on Numerical Analysis*, **54**(5):2951–2973, 2016. https://doi.org/10.1137/15M1050082.

[23] B. Rivière, J. Tan and T. Thompson. Error analysis of primal discontinuous Galerkin methods for a mixed formulation of the Biot equations. *Computers & Mathematics with Applications. An International Journal*, **73**(4):666–683, 2017. https://doi.org/10.1016/j.camwa.2016.12.030.

[24] A. Sreekumar, S. P. Triantafyllou, F.-X. Bécot and F. Chevillotte. Multiscale VEM for the Biot consolidation analysis of complex and highly heterogeneous domains. *Computer Methods in Applied Mechanics and Engineering*, **375**:113543, 2021. https://doi.org/10.1016/j.cma.2020.113543.

[25] A. Sreekumar, S. P. Triantafyllou and F. Chevillotte. Virtual elements for sound propagation in complex poroelastic media. *Computational Mechanics*, **144**:347–382, 2021. https://doi.org/10.1007/s00466-021-02078-2.

[26] X. Tang, Z. Liu, B. Zhang and M. Feng. On the locking-free three-field virtual element methods for Biot's consolidation model in poroelasticity. *ESAIM. Mathematical Modelling and Numerical Analysis*, **55**:S909–S939, 2021. https://doi.org/10.1051/m2an/2020064.

[27] G. Vacca and L.B. da Veiga. Virtual element methods for parabolic problems on polygonal meshes. *Numerical Methods for Partial Differential Equations. An International Journal*, **31**(6):2110–2134, 2015. https://doi.org/10.1002/num.21982.

[28] N. Verma, B. Gómez-Vargas, L. Miguel De Oliveira Vilaca, S. Kumar and R. Ruiz-Baier. Well-posedness and discrete analysis for advection-diffusion-reaction in poroelastic media. *Applicable Analysis*, **101**(14):4914–4941, 2022. https://doi.org/10.1080/00036811.2021.1877677.

[29] N. Verma and S. Kumar. Lowest order virtual element approximations for transient Stokes problem on polygonal meshes. *Calcolo*, **58**(48), 2021. https://doi.org/10.1007/s10092-021-00440-7.

[30] S.-Y. Yi. A study of two modes of locking in poroelasticity. *SIAM Journal on Numerical Analysis*, **55**(4):1915–1936, 2017. https://doi.org/10.1137/16M1056109.