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Voronovskaya Type Results and Operators Fixing Two Functions

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Abstract. The present paper deals with positive linear operators which fix two functions. The transfer of a given sequence (L_n) of positive linear operators to a new sequence (K_n) is investigated. A general procedure to construct sequences of positive linear operators fixing two functions which form an Extended Complete Chebyshev system is described. The Voronovskaya type formula corresponding to the new sequence which is strongly influenced by the nature of the fixed functions is obtained. In the last section our results are compared with other results existing in literature.

Keywords: positive linear operators, Voronovskaya type theorem, extended complete Chebyshev system, operators fixing two functions.

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1 Introduction

Let $I \subseteq \mathbb{R}$ be an interval and $C(I)$ the space of all continuous, real-valued functions defined on I. Let $L_n : D \to C(I)$, $n \geq 1$, be a sequence of positive linear operators, where D is a linear subspace of $C(I)$. In many situations, the sequence $(L_n)_{n\geq 1}$ satisfies a Voronovskaya type formula, i.e.,

$$\lim_{n\to\infty} n\left(L_n(f;x) - f(x)\right) = \left(a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)\right) f(x),$$

where $f \in D \cap C^2(I)$, $x \in I$.

Such formulas are essential tools in approximation by positive linear operators. They are used to describe the rates of convergence and the saturation class. The iterates of certain positive linear operators can be used in order to approximate C_0-semigroups of operators; in this case the Voronovskaya operator and the infinitesimal generator of the semigroup are strongly related (see [11,12]).

Usually one uses operators which preserve the constant functions; then $c(x) = 0$, $x \in I$. If, in addition, $L_n v = v$, $n \geq 1$, for a non-constant function $v \in C^2(I)$, then $a(x)$ and $b(x)$ are related by $a(x)v''(x) + b(x)v'(x) = 0$, $x \in I$.

In the last years, several papers were published, dealing with positive linear operators which fix two functions. A starting point was the paper [27] written by P.J. King who constructed positive linear operators on $C[0,1]$ fixing the constant function 1 and the function x^2. Many generalizations followed, dealing with operators fixing 1 and a given function τ; see Section 6. Recently, operators fixing two exponential functions were constructed: see [2,3,4,7,14,23].

The aim of our paper is twofold. On one hand, we investigate the transfer of a given sequence (L_n) of positive linear operators from an interval I to another interval J, and describe the Voronovskaya type formula corresponding to the new sequence (K_n). The properties of (K_n) inherited from (L_n), are also considered, as well as the inverse transfer from (K_n) to (L_n).

On the other hand, we propose a general procedure to construct sequences of positive linear operators fixing two functions which form an Extended Complete Chebyshev system. As mentioned above, the structure of the Voronovskaya operator is strongly influenced by the nature of the fixed functions. In a certain sense, the quality of the approximation offered by (L_n) can be expressed in terms of the corresponding Voronovskaya operator. We compare our results, from this point of view, with other results existing in the literature.

The basic definitions (in particular, the transfer of operators) are presented at the end of this section. Section 2 is devoted to the inverse transfer and the inherited properties. The Voronovskaya formula for the new operators is established in Section 3. Operators A_n fixing two functions are constructed in Section 4. In Section 5, the convexity with respect to these two functions is characterized in terms of the operators A_n and in terms of their Voronovskaya operator. Section 6 is devoted to examples and applications illustrating our general results and surveying some previous results existing in the literature.

We use notations usual in approximation theory by positive linear operators. As far as the domains of operators are concerned, we consider the maximal ones, i.e., we let the operators to act on the functions for which the involved series or integrals are convergent. We end this section with some basic definitions and notations.

Let I and J be intervals and $D \subseteq C(I)$ a linear subspace containing the polynomial functions. By a slight abuse of notation we denote by e_i, $i = 0, 1, \dots,$ the function $e_i(t) = t^i$ defined either on I or on J.

Let $L_n : D \to C(I)$, $n \geq 1$, be positive linear operators such that

$$L_n e_0 = e_0, \ n \geq 1. \tag{1.1}$$

Consider two continuous functions, $u : I \to J$, $v : J \to I$, such that

$$u(v(t)) = t, \ t \in J. \tag{1.2}$$

Then v is injective, hence strictly monotone on J.

Let $D_1 := \{g \in C(J) : g \circ u \in D\}$. Consider the operators $K_n : D_1 \to C(J)$,

$$K_n(g; t) := L_n(g \circ u; v(t)), \, g \in D_1, \, t \in J. \tag{1.3}$$

By using (1.1) and (1.2), we see that

i) $K_n e_0 = e_0$, $n \geq 1$,

ii) If $\lim\limits_{n\to\infty} L_n(f; x) = f(x), \ f \in D, \, x \in I$, then

$$\lim_{n\to\infty} K_n(g; t) = g(t), g \in D_1, \, t \in J.$$

2 Inverse transfer, iterates, commutativity

With notation from the preceding section, let $H := \{h \in D : h \circ v \circ u = h\}$.

Theorem 1. *Suppose that $L_n(H) \subset H$, $n \geq 1$. Then*

i) $L_n h = (K_n(h \circ v)) \circ u$, $h \in H$.

ii) $K_n^k g = \big(L_n^k(g \circ u)\big) \circ v$, $g \in D_1$, $k \in \mathbb{N}$.

iii) If $L_n L_m = L_m L_n$ on H, then $K_n K_m = K_m K_n$ on D_1.

Proof. i) Let $h \in H$, $x \in I$. Then $L_n h \in H$, hence $L_n h = (L_n h) \circ v \circ u$. Now we have

$$L_n h(x) = (L_n h) \, (v(u(x)) = (L_n(h \circ v \circ u)) \, (v(u(x))) = K_n \, (h \circ v) \, (u(x)),$$

and this proves i).

ii) Let us prove that

$$g \circ u \in H, \ \text{for all } g \in D_1. \tag{2.1}$$

Indeed, if $g \in D_1$, then $g \circ u \in D$. Moreover, $u \circ v$ is the identity function on J, so that $g \circ u = (g \circ u) \circ v \circ u$, which proves (2.1). Since $L_n(H) \subset H$, we have also from (2.1):

$$L_n^k(g \circ u) \in H, \ \text{for all } g \in D_1, \, k \in \mathbb{N}. \tag{2.2}$$

For $k=1$, ii) is obviously true. Suppose that ii) is true for a certain k. Then, for $g \in D_1$,

$$K_n^{k+1} g = K_n\left(K_n^k g\right) = K_n\left(L_n^k(g\circ u)\circ v\right) = \left(L_n\left(\left(L_n^k(g\circ u)\circ v\right)\circ u\right)\right)\circ v.$$

According to (2.2), $L_n^k(g\circ u)\circ v\circ u = L_n^k(g\circ u)$, so that

$$K_n^{k+1} g = L_n\left(L_n^k(g\circ u)\right)\circ v = \left(L_n^{k+1}(g\circ u)\right)\circ v.$$

This induction argument proves ii).
iii) Suppose that $L_nL_m = L_mL_n$ on H. Let $g \in D_1$. Then

$$K_nK_m g = \left(L_n\left((K_m g)\circ u\right)\right)\circ v = \left(L_n\left(\left(\left(L_m(g\circ u)\right)\circ v\right)\right)\circ u\right)\circ v.$$

Using again (2.2) we see that $L_m(g\circ u) = (L_m(g\circ u))\circ v\circ u$, which leads to

$$K_nK_m g = \left(L_n(L_m(g\circ u))\right)\circ v = \left(L_nL_m(g\circ u)\right)\circ v. \tag{2.3}$$

Similarly,

$$K_mK_n g = \left(L_mL_n(g\circ u)\right)\circ v. \tag{2.4}$$

From (2.3) and (2.4) we infer that $K_nK_mg = K_mK_ng$, $g\in D_1$. $\square$

3 Transfer of Voronovskaya formula

In addition to the preceding hypotheses, in this section we suppose that $u \in C^2(I)$, $v \in C^2(J)$, and the sequence (L_n) satisfies the following Voronovskaya type formula:

$$\lim_{n\to\infty} n\left(L_n(f;x) - f(x)\right) = \alpha(x)f''(x) + \beta(x)f'(x), \tag{3.1}$$

for all $x \in I$ and $f \in D\cap C^2(I)$, where $\alpha,\beta \in C(I)$ are two given functions.

Theorem 2. *If $g\in C^2(J)$ and $t\in J$, such that $g\circ u \in D$ and $v'(t)\neq 0$, then*

$$\lim_{n\to\infty} n\left(K_n(g;t) - g(t)\right) = \frac{\alpha(v(t))}{v'(t)}\left(\frac{g'}{v'}\right)'(t) + \beta(v(t))\left(\frac{g'}{v'}\right)(t). \tag{3.2}$$

Proof. Combining (1.3) and (3.1), we get

$$\begin{aligned}\lim_{n\to\infty} n\left(K_n(g;t) - g(t)\right) &= \lim_{n\to\infty} n\left(L_n(g\circ u; v(t)) - (g\circ u)(v(t))\right)\\ &= \alpha(v(t))(g\circ u)''(v(t)) + \beta(v(t))(g\circ u)'(v(t)).\end{aligned} \tag{3.3}$$

Recall that $u(v(t)) = t,\ t\in J$; this implies $u'(v(t)) = \frac{1}{v'(t)}$, and from

$$(g\circ u)'(v(t)) = g'(u(v(t)))u'(v(t)), \tag{3.4}$$

we obtain

$$(g\circ u)'(v(t)) = g'(t)/v'(t). \tag{3.5}$$

Furthermore, (3.4) leads to

$$\begin{aligned}(g \circ u)''(v(t)) &= g''(u(v(t)))\left(u'(v(t))\right)^2 + g'(u(v(t)))u''(v(t)) \\ &= g''(t)\frac{1}{(v'(t))^2} + g'(t)u''(v(t)).\end{aligned} \tag{3.6}$$

Using again $u'(v(t))v'(t) = 1$ we get $u''(v(t))(v'(t))^2 + u'(v(t))v''(t) = 0$, i.e.,

$$u''(v(t)) = -v''(t)/(v'(t))^3. \tag{3.7}$$

Now (3.6) and (3.7) yield

$$(g \circ u)''(v(t)) = \frac{g''(t)}{(v'(t))^2} - \frac{g'(t)}{(v'(t))^3}v''(t) = \frac{1}{v'(t)}\left(\frac{g'}{v'}\right)'(t). \tag{3.8}$$

From (3.3), (3.5) and (3.8) we get (3.2), and this concludes the proof. □

4 Operators fixing two functions

Let $\{u_0, u_1\}$ be an ECT-system on $[a, b]$ (see [26, Chapter 11]). Then, according to [26, Theorem 11.1.2],

$$u_0(t) = w_0(t),\ u_1(t) = w_0(t)\int_a^t w_1(s)ds,\ t \in [a, b],$$

for some strictly positive functions $w_0 \in C^1[a, b]$, $w_1 \in C[a, b]$.

In this section we are concerned with positive linear operators fixing the two-dimensional linear subspace generated by an ECT-system. More precisely, consider the following setting. Let I be an interval, $\tau, \gamma \in C(I)$, $\gamma : I \to I$ bijective, $\tau(x) > 0$, $x \in I$. Let $L_n : D \to C(I)$, $n \geq 1$, be positive linear operators, where D is a linear subspace of $C(I)$ containing the polynomial functions. Suppose that

$$L_n e_0 = e_0,\ \ L_n e_1 = e_1,\ \ n \geq 1. \tag{4.1}$$

Let $D_2 := \left\{f \in C(I) : \frac{f}{\tau} \circ \gamma^{-1} \in D\right\}$. Consider the positive linear operators $A_n : D_2 \to C(I)$,

$$A_n(f; x) = \tau(x)L_n\left(\frac{f}{\tau} \circ \gamma^{-1}; \gamma(x)\right),\ \ f \in D_2,\ x \in I. \tag{4.2}$$

Then we have

$$A_n\tau = \tau,\ A_n(\gamma\tau) = \gamma\tau,\ \ n \geq 1. \tag{4.3}$$

Remark 1. Suppose that $I = [a, b]$, $\gamma \in C^1(I)$, $\gamma'(x) > 0$, $x \in I$. Take $w_0 = \tau$, $w_1 = \gamma'$. Then $u_0 = \tau$, $u_1 = \tau(\gamma - \gamma(a))$, and therefore $\{\tau, \tau(\gamma - \gamma(a))\}$ is an ECT-system. The linear subspace generated by it is the same as the linear subspace generated by $\{\tau, \gamma\tau\}$.

In order to present the Voronovskaya formula for the sequence (A_n) we need some additional hypotheses. In fact, we suppose that

$$\tau, \gamma \in C^2(I),\ \gamma^{-1} \in C^2(I), \tag{4.4}$$

$$\tau(x) > 0,\ \gamma'(x) > 0,\ x \in I. \tag{4.5}$$

Moreover, assume that

$$\lim_{n\to\infty} n(L_n(h;x) - h(x)) = \alpha(x)h''(x),\ x \in I,\ \text{for all } h \in D \cap C^2(I). \tag{4.6}$$

Remark 2. Due to (4.1), we have $L_n f \geq f$ for all convex functions $f \in D$. In particular, $L_n e_2 \geq e_2$, and (4.6) leads to

$$\alpha(x) \geq 0,\ x \in I. \tag{4.7}$$

Theorem 3. *Under the above assumptions, if $f \in D_2 \cap C^2(I)$ and $x \in I$, then*

$$\lim_{n\to\infty} n\left(A_n(f;x) - f(x)\right) = \tau(x)\frac{\alpha(\gamma(x))}{\gamma'(x)}\left(\frac{(f/\tau)'}{\gamma'}\right)'(x). \tag{4.8}$$

Proof. Choosing $g = \dfrac{f}{\tau}$, $u = \gamma^{-1}$, $v = \gamma$, and using (1.3), (4.2), we get

$$A_n(f;x) = \tau(x)L_n\left(\frac{f}{\tau}\circ\gamma^{-1};\gamma(x)\right) = \tau(x)K_n(g;x).$$

Due to (4.6), we can apply Theorem 2 with $\beta = 0$. It follows that

$$\lim_{n\to\infty} n(A_n(f;x) - f(x)) = \lim_{n\to\infty} n\left(\tau(x)K_n\left(\frac{f}{\tau};x\right) - f(x)\right)$$

$$= \tau(x)\lim_{n\to\infty} n\left(K_n\left(\frac{f}{\tau};x\right) - \frac{f}{\tau}(x)\right) = \tau(x)\frac{\alpha(\gamma(x))}{\gamma'(x)}\left(\frac{(f/\tau)'}{\gamma'}\right)'(x),$$

and the proof is finished. □

5 Convex functions with respect to $\{\tau, \gamma\tau\}$

Suppose that τ and γ satisfy (4.4) and (4.5). A function $f \in C^2(I)$ is called convex with respect to $\{\tau, \gamma\tau\}$ if

$$\Delta_f := \begin{vmatrix} f & f' & f'' \\ \tau & \tau' & \tau'' \\ \gamma\tau & (\gamma\tau)' & (\gamma\tau)'' \end{vmatrix} \geq 0 \text{ on } I.$$

On the other hand, consider the differential operator

$$Af(x) := \frac{d}{dx}\left(\frac{1}{\gamma'(x)}\frac{d}{dx}\frac{f(x)}{\tau(x)}\right).$$

It is easy to verify that

$$\begin{aligned} Af &= \frac{1}{\tau^3 {\gamma'}^2} \Delta_f \\ &= \frac{1}{\tau^3 {\gamma'}^2} \left(\tau^2 \gamma' f'' - (\tau^2 \gamma'' + 2\tau\tau'\gamma') f' + \left(\tau\tau'\gamma'' + 2{\tau'}^2\gamma' - \tau\gamma'\tau'' \right) f \right). \end{aligned}$$

Consequently, we have

Proposition 1. *$f \in C^2(I)$ is convex with respect to $\{\tau, \gamma\tau\}$ if and only if $Af \geq 0$ on I.*

Moreover, Theorem 2 shows that if $f \in D_2 \cap C^2(I)$ and $x \in I$, then

$$\lim_{n\to\infty} n \left(A_n(f;x) - f(x) \right) = \tau(x) \frac{\alpha(\gamma(x))}{\gamma'(x)} Af(x). \tag{5.1}$$

According to (4.7), $\alpha(x) \geq 0$, $x \in I$. Suppose that this inequality is strict at each interior point of I. Now we can prove

Theorem 4. *$f \in C^2(I) \cap D_2$ is convex with respect to $\{\tau, \gamma\tau\}$ iff $A_n f \geq f$, $n \geq 1$.*

Proof. (5.1) shows that if $A_n f \geq f$ on I, then $Af \geq 0$, and so f is convex with respect to $\{\tau, \gamma\tau\}$. Conversely, suppose that $Af(x) \geq 0$, $x \in I$. Let

$$\varphi(t,x) := \tau(t)(\gamma(t) - \gamma(x)),\ t, x \in I.$$

Since γ is strictly increasing on I, we have $\varphi(t,x) > 0$ if $t > x$, and $\varphi(t,x) < 0$ if $t < x$.

Let $a \in I$. The above inequalities show that

$$\int_a^t \varphi(t,x) Af(x) dx \geq 0,\ t \in I. \tag{5.2}$$

On the other hand,

$$\begin{aligned} &\int_a^t \varphi(t,x) Af(x) dx = \tau(t) \int_a^t (\gamma(t) - \gamma(x)) \frac{d}{dx} \left(\frac{1}{\gamma'(x)} \frac{d}{dx} \frac{f(x)}{\tau(x)} \right) dx \\ &= \tau(t) \left((\gamma(t) - \gamma(x)) \frac{1}{\gamma'(x)} \frac{d}{dx} \frac{f(x)}{\tau(x)} \right) \Bigg|_{x=a}^{x=t} + \tau(t) \int_a^t \left(\frac{d}{dx} \frac{f(x)}{\tau(x)} \right) dx \\ &= -\tau(t) \left((\gamma(t) - \gamma(a)) \frac{1}{\gamma'(a)} \left(\frac{f}{\tau} \right)' (a) \right) + \tau(t) \left(\frac{f(t)}{\tau(t)} - \frac{f(a)}{\tau(a)} \right) \\ &= f(t) - \frac{f(a)}{\tau(a)} \tau(t) - \frac{1}{\gamma'(a)} \left(\frac{f}{\tau} \right)' (a) \tau(t)(\gamma(t) - \gamma(a)). \end{aligned}$$

Combined with (5.2), this yields

$$f \geq \frac{f(a)}{\tau(a)} \tau + \frac{1}{\gamma'(a)} \left(\frac{f}{\tau} \right)' (a)(\gamma\tau - \gamma(a)\tau).$$

Now using (4.3), we get

$$A_n f \geq \frac{f(a)}{\tau(a)}\tau + \frac{1}{\gamma'(a)}\left(\frac{f}{\tau}\right)'(a)(\gamma\tau - \gamma(a)\tau), \; n \geq 1.$$

In particular, $A_n f(a) \geq f(a)$, $n \geq 1$, for each $a \in I$. This means that $A_n f \geq f$, and the proof is complete. □

Remark 3. i) Let p be a polynomial function. Then

$$A_n\left((p \circ \gamma)\tau; x\right) = \tau(x) L_n\left(p; \gamma(x)\right),$$

and in many cases we can compute $L_n(p; \cdot)$.
ii) Denote $\varphi_n := L_n e_2 - e_2$. Then $|L_n(f;x) - f(x)| \leq \Psi_n(x)\omega(f; \sqrt{\varphi_n(x)})$ for a suitable function $\Psi_n(x)$. Consequently,

$$|A_n(f;x) - f(x)| \leq \tau(x)\Psi_n(x)\omega\left(\frac{f}{\tau} \circ \gamma^{-1}; \sqrt{\varphi_n(x)}\right).$$

iii) If $f \in D_2$ and $\dfrac{f}{\tau} \circ \gamma^{-1}$ is convex, then

$$L_n\left(\frac{f}{\tau} \circ \gamma^{-1}; \gamma(x)\right) \geq \left(\frac{f}{\tau} \circ \gamma^{-1}\right)(\gamma(x)),$$

and consequently $A_n(f;x) \geq f(x)$, $x \in I$.

6 Examples and applications

In this section we present some examples illustrating the preceding general results.

Example 1. Consider the intervals $I = [0,1)$, $J = [0,\infty)$, and the functions $u : [0,1) \to [0,\infty)$, $u(x) = \frac{x}{1-x}$; $v : [0,\infty) \to [0,1)$, $v(t) = \frac{t}{1+t}$. They satisfy the condition (1.2).

Let $L_n : C[0,1] \to C[0,1]$,

$$L_n(f;x) = \begin{cases} (1-x)^{n+1}\displaystyle\sum_{k=0}^{\infty}\binom{n+k}{k}x^k f\left(\frac{k}{n+k}\right), \; x \in [0,1), \\ f(1), \; x = 1, \end{cases}$$

be the Meyer-König and Zeller operators (see [11, 28]). It is well known that $L_n e_0 = e_0$ and $L_n e_1 = e_1$.

In this context the operators K_n introduced by (1.3) are described as

$$K_n(g;t) = \frac{1}{(1+t)^{n+1}}\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{t^k}{(1+t)^k} g\left(\frac{k}{n}\right), \; g \in D_1, \; t \in [0,+\infty), \tag{6.1}$$

where $D_1 := \{g \in C[0,\infty) | \exists\, g(\infty) := \lim_{t\to\infty} g(t) \in \mathbb{R}\}$. The Voronovskaya operator for the sequence (L_n) is $\frac{x(1-x)^2}{2}\frac{d^2}{dx^2}$ (see [33]). According to Theorem 3.1, the Voronovskaya operator for (K_n) is $\frac{t(1+t)}{2}\frac{d^2}{dt^2} + t\frac{d}{dt}$.

As mentioned above, condition (4.1) is satisfied. Choosing in Section 4 $\tau = e_0$ and $\gamma = v$, we have $\gamma^{-1} = u$. Consequently, the operators K_n can be described as in (4.2), and we see that they fix the functions e_0 and v (see (4.3)).

Concerning the asymptotic behavior of the iterates of L_n, it is known (see [34]) that

$$\lim_{k\to\infty} L_n^k(f;x) = (1-x)f(0) + xf(1),\ f \in C[0,1] \tag{6.2}$$

uniformly for $x \in [0,1]$. According to Theorem 1 (ii),

$$\begin{aligned}\lim_{k\to\infty} K_n^k(g;t) &= \lim_{k\to\infty} L_n^k(g\circ u; v(t)) \\ &= (1-v(t))g(u(0)) + v(t)g(u(1)) = \frac{1}{1+t}\left(g(0)+tg(\infty)\right)\end{aligned} \tag{6.3}$$

uniformly for $t \in [0,\infty)$.

Remark 4. The operators K_n given by (6.1) should be compared with the classical Baskakov operators

$$H_n(g;t) := \frac{1}{(1+t)^n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{t^k}{(1+t)^k}g\left(\frac{k}{n}\right),\ t \geq 0.$$

These operators fix the functions e_0 and e_1. A way to represent the MKZ operators in terms of Baskakov operators can be found in [9]. The MKZ operators L_n can be represented in terms of the operators K_n defined by (6.1) if we use Theorem 1 (i).

Starting with the Baskakov operators H_n, let us construct the operators $\tilde{K}_n(f;x) := H_n(f\circ v; u(x))$. Then

$$\tilde{K}_n(f;x) = (1-x)^n\sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k f\left(\frac{k}{n+k}\right),\ x \in [0,1)$$

with Voronovskaya operator $\frac{x(1-x)^2}{2}\frac{d^2}{dx^2} - x(1-x)\frac{d}{dx}$. They fix the functions e_0 and u.

Example 2. Let L_n be the classical Bernstein operators on $C[0,1]$, i.e.,

$$L_n(f;x) = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right),\ f \in C[0,1],\ x \in [0,1]. \tag{6.4}$$

With I, J, u, v, D_1 as in Example 1, the corresponding operators K_n will be

$$K_n(f;t) = \sum_{k=0}^{n}\binom{n}{k}\frac{t^k}{(1+t)^n}g\left(\frac{k}{n-k}\right),\ g \in D_1, t \geq 0. \tag{6.5}$$

In the above sum, the last value of g is $g(\infty)$.

The Voronovskaya operator for the Bernstein operators is $\frac{x(1-x)}{2}\frac{d^2}{dx^2}$. Theorem 2 shows that the corresponding Voronovskaya operator for the sequence $(K_n)_{n\geq 1}$ is $\frac{t(1+t)^2}{2}\frac{d^2}{dt^2}+t(1+t)\frac{d}{dt}$.

The operators K_n fix the functions e_0 and v. The asymptotic behavior of the iterates of L_n from (6.4), respectively K_n from (6.5) is the same as in (6.2), (6.3).

Remark 5. The operators K_n from (6.5) should be compared with the classical Bleimann-Butzer-Hahn operators

$$G_n(g;t)=\sum_{k=0}^{n}\binom{n}{k}\frac{t^k}{(1+t)^n}g\left(\frac{k}{n-k+1}\right),\ g\in C[0,\infty),\ t\geq 0.$$

A representation of the BBH operators in terms of the Bernstein operators can be found in [9]. Starting with the BBH operators G_n, let us construct the operators $\tilde{K}_n(f;x):=G_n(f\circ v;u(x))$. Then

$$\tilde{K}_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n+1}\right),f\in C[0,1],x\in[0,1]$$

with Voronovskaya operator $\frac{x(1-x)}{2}\frac{d^2}{dx^2}-x\frac{d}{dx}$. They fix e_0 and u.

Example 3. Consider the Baskakov-Durrmeyer operators

$$L_n(f;x)=n\sum_{k=0}^{\infty}\binom{n+k}{k}^2\frac{x^k}{(1+x)^{n+k+1}}\int_0^{\infty}\frac{t^kf(t)}{(1+t)^{n+k+1}}dt,\ x\geq 0.$$

For details, see [24, p.13, (2.15)] with $c=1$. With $u:[0,\infty)\to[0,1)$, $u(x)=\frac{x}{1+x}$, and $v:[0,1)\to[0,+\infty)$, $v(t)=\frac{t}{1-t}$, we get

$$K_n(g;t)=n(1-t)^{n+1}\sum_{k=0}^{\infty}\binom{n+k}{k}^2t^k\int_0^1 s^k(1-s)^{n-1}g(s)ds.$$

It is known that $L_nL_m=L_mL_n$, $m,n\geq 1$ (see [24, p.14]). Since $(v\circ u)(x)=x$, we can apply Theorem 1 (iii) to conclude that $K_nK_m=K_mK_n$, $m,n\geq 1$.

Example 4. Let $L_n:C_{2\pi}[-\pi,\pi]\to C_{2\pi}[-\pi,\pi]$ be the De la Vallée Poussin operators, i.e.,

$$L_n(f;x)=\frac{1}{2\pi}\frac{(n!)^2}{(2n)!}4^n\int_{-\pi}^{\pi}f(s)\left(\cos\frac{x-s}{2}\right)^{2n}ds,\ f\in C_{2\pi}[-\pi,\pi].$$

With $u:[-\pi,\pi]\to[0,1]$, $u(x)=\sin^2\dfrac{x}{2}$, and $v:[0,1]\to[-\pi,\pi]$, $v(t)=\arccos(1-2t)$, the operators K_n defined by (1.3) are the Durrmeyer operators with Chebyshev weight $w(t)=t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}$, i.e., $K_n:C[0,1]\to C[0,1]$, (see [20])

$$K_n(g;t)=\sum_{j=0}^{n}\binom{n}{j}t^j(1-t)^{n-j}\frac{\int_0^1 s^{j-\frac{1}{2}}(1-s)^{n-j-\frac{1}{2}}g(s)ds}{\int_0^1 s^{j-\frac{1}{2}}(1-s)^{n-j-\frac{1}{2}}ds}.$$

It is elementary to prove that

$$H := \{h \in C_{2\pi}[-\pi,\pi] |\, h \circ v \circ u = h\} = \{h \in C_{2\pi}[-\pi,\pi] |\, h \text{ is even}\},$$

and $L_n(H) \subset H$. Moreover, $L_n L_m = L_m L_n$ on H. Therefore, all the conclusions i), ii), iii) of Theorem 1 are valid in this context. For details and several applications, see [8, Section 5].

Example 5. Let $L_n : D \to C(\mathbb{R})$, $L_n = W_{\frac{1}{4n}} f$, $f \in D$, where

$$W_t(f;x) := \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(s+x) e^{-\frac{s^2}{4t}} ds,\ t > 0,\ x \in \mathbb{R}$$

are the Weierstrass operators (see [11]). The domain D of L_n contains, in particular, the bounded continuous functions and the polynomial functions defined on $\mathbb{R}$.

Let $u : \mathbb{R} \to [0,1]$, $u(x) = \sin^2 x$, and $v : [0,1] \to \mathbb{R}$, $v(t) = \arcsin \sqrt{t}$. Then the operators $K_n : C[0,1] \to C[0,1]$ defined by (1.3) are described as follows:

$$K_n(g;t) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} g\left(\sin^2(s + \arcsin\sqrt{t})\right) e^{-ns^2} ds,\ g \in C[0,1],\ t \in [0,1].$$

These operators K_n were studied in [13, p.23].

Using [11, Section 5.2.9] we find that

$$\lim_{t \to 0} \frac{1}{t} \left(W_t(f;x) - f(x)\right) = f''(x),\ \ x \in \mathbb{R},\ f \in D \cap C^2(\mathbb{R}).$$

Therefore,

$$\lim_{n\to\infty} n\left(L_n(f;x) - f(x)\right) = \frac{1}{4} f''(x),\ \ x \in \mathbb{R},\ \ f \in D \cap C^2(\mathbb{R}).$$

Now Theorem 2 can be applied with $\alpha(x) = \frac{1}{4}$, $\beta(x) = 0$, yielding

$$\lim_{n\to\infty} n\left(K_n(g;t) - g(t)\right) = t(1-t)g''(t) + \left(\frac{1}{2} - t\right) g'(t),\ \ g \in C^2[0,1],\ t \in [0,1].$$

This result was obtained with different methods in [13, p.23].

It is elementary to prove that

$$H := \{h \in C(\mathbb{R}) |\, h \circ v \circ u = h\} = \{h \in C_\pi(\mathbb{R}) |\, h \text{ is even}\},$$

and $L_n(H) \subset H$. Moreover, $L_n L_m = L_m L_n$ on H, $m,n \geq 1$. Consequently, all the conclusions i), ii), iii) of Theorem 1 are valid in this setting.

Remark 6. Consider two sequences of positive linear operators $(U_{n,j})_{n\geq 1}$, $j = 1,2$, on the same interval I. Suppose that they fix the same two functions w_1, w_2:

$$U_{n,j} w_i = w_i,\ n \geq 1,\ i,j = 1,2. \tag{6.6}$$

Moreover, suppose that each sequence satisfies a Voronovskaya type formula. Due to (6.6), these formulas are of the form

$$\lim_{n\to\infty} n\left(U_{n,j}f(x)-f(x)\right)=\omega_j(x)\left(\frac{d^2}{dx^2}+p(x)\frac{d}{dx}+q(x)\right)f(x),$$

where the functions $p(x)$ and $q(x)$ satisfy the system

$$w_i''(x)+p(x)w_i'(x)+q(x)w_i(x)=0,\ i=1,2.$$

Thus we have

$$\lim_{n\to\infty} n\left|U_{n,j}f(x)-f(x)\right|=|\omega_j(x)|\left|\left(\frac{d^2}{dx^2}+p(x)\frac{d}{dx}+q(x)\right)f(x)\right|.$$

So, in this sense, if on a certain subinterval I_0 we have $|\omega_1(x)|\le|\omega_2(x)|$, $x\in I_0$, then the quality of the approximation on I_0 offered by $(U_{n,1})_{n\ge1}$ is better than offered by $(U_{n,2})_{n\ge1}$.

Example 6. Consider the operators (P_n) on $[0,\infty)$ constructed in [23]. They preserve the functions e^{ax} and e^{bx}; the corresponding Voronovskaya operator is (see [23, (11), p.5080]):

$$x\left(\frac{d^2}{dx^2}-(a+b)\frac{d}{dx}+ab\right).$$

Let $a<b$, $\tau(x):=e^{ax}$, $\gamma(x):=e^{(b-a)x}$, $x\ge0$. Taking the Szász-Mirakyan operators in the role of L_n, let us construct the operators A_n as in (4.2). They fix the same functions $e^{ax}=\tau(x)$, $e^{bx}=\gamma(x)\tau(x)$; see (4.3). The Voronovskaya operator (see (4.8)) is

$$\frac{e^{(a-b)x}}{2(b-a)^2}\left(\frac{d^2}{dx^2}-(a+b)\frac{d}{dx}+ab\right).$$

So, we have to compare the functions $\omega_1(x)=x$ and $\omega_2(x)=\frac{1}{2(b-a)^2}e^{(a-b)x}$, $x\in[0,\infty)$. It is easy to see that there exists $x_0\in(0,\infty)$ such that $\omega_2(x)>\omega_1(x)$ for $x<x_0$, and $\omega_2(x)<\omega_1(x)$ for $x>x_0$. Moreover, $\lim\limits_{x\to\infty}\omega_1(x)=\infty$ and $\lim\limits_{x\to\infty}\omega_2(x)=0$. We conclude that, in the sense of Remark 6, the approximation offered by $(A_n)_{n\ge1}$ on (x_0,∞) is better than that offered by $(P_n)_{n\ge1}$.

Example 7. P.J. King (see [27]) constructed a sequence of positive linear operators on C[0,1] fixing e_0 and e_2. The coresponding Voronovskaya operator was described in [22]; it is

$$\frac{x(1-x)}{2}\frac{d^2}{dx^2}-\frac{1-x}{2}\frac{d}{dx}.$$

Example 8. In order to generalize the preceding example, let $\tau\in C[0,1]$ be strictly increasing, $\tau(0)=0$, $\tau(1)=1$, and let B_n be the classical Bernstein operators on $C[0,1]$. Then the operators $V_n:C[0,1]\to C[0,1]$,

$$V_nf:=(B_nf)\circ(B_n\tau)^{-1}\circ\tau,\ f\in C[0,1]$$

fix the functions e_0 and τ. Moreover, if $\tau \in C^2[0,1]$ and $\tau'(x) > 0$, $x \in [0,1]$, then the Voronovskaya operator is

$$\frac{x(1-x)}{2}\left(\frac{d^2}{dx^2} - \frac{\tau''(x)}{\tau'(x)}\frac{d}{dx}\right).$$

For details and other applications, see [22].

Example 9. Another generalization of Example 7 can be found in [10]. The authors of that article constructed a sequence of positive linear operators on $C[0,1]$ fixing e_0 and e_j, for a fixed integer $j \geq 1$. It was conjectured in [18] that the corresponding Voronovskaya operator is

$$\frac{x(1-x)}{2}\frac{d^2}{dx^2} - (j-1)\frac{1-x}{2}\frac{d}{dx}.$$

This conjecture was validated by M. Birou [15].

Example 10. Another sequence of positive linear operators fixing e_0 and e_j was introduced in [16]. It has the Voronovskaya operator

$$x(1-x)\frac{d^2}{dx^2} - (j-1)(1-x)\frac{d}{dx}.$$

Example 11. Let again $\tau \in C^2[0,1]$, $\tau(0) = 0$, $\tau(1) = 1$, $\tau'(x) > 0$, $x \in [0,1]$, and let B_n be the Bernstein operators on $C[0,1]$. The operators $B_n^\tau : C[0,1] \to C[0,1]$,

$$B_n^\tau f := \big(B_n(f \circ \tau^{-1})\big) \circ \tau, \;\; f \in C[0,1] \tag{6.7}$$

were considered in [17].

They fix e_0 and τ. It was proved there that the associated Voronovskaya operator is

$$\frac{\tau(1-\tau)}{2}\left(\frac{1}{\tau'^2}\frac{d^2}{dx^2} - \frac{1}{\tau'^3}\frac{d}{dx}\right).$$

Clearly, the sequence $(B_n^\tau)_{n\geq 1}$ is a particular case of the sequence $(K_n)_{n\geq 1}$ from (1.3). If we take $\tau = e_j$ on $(0,1]$, we get the operator

$$\frac{x^{1-j}(1-x^j)}{2j^2}\left(x\frac{d^2}{dx^2} - (j-1)\frac{d}{dx}\right),$$

which should be compared in the sense of Remark 6 with the operator from Example 9, i.e.,

$$\frac{1-x}{2}\left(x\frac{d^2}{dx^2} - (j-1)\frac{d}{dx}\right).$$

More precisely, we have to solve the inequation

$$\frac{x^{1-j}(1-x^j)}{2j^2} \leq \frac{1-x}{2}, \text{ with } x \in (0,1]. \tag{6.8}$$

It is easy to see that there exists $x_j \in (0,1)$ such that (6.8) is satisfied if and only if $x \in [x_j, 1]$.

Example 12. Consider the operators defined in (6.7) with $\tau(x) = \sqrt{x}$, i.e., $B_n(f(t^2); \sqrt{x})$. They were investigated in [19]. As in Example 11, we find that the Voronovskaya operator is

$$2x\sqrt{x}(1-\sqrt{x})\frac{d^2}{dx^2} + \sqrt{x}(1-\sqrt{x})\frac{d}{dx}.$$

Clearly, these operators fix e_0 and $\tau(x) = \sqrt{x}$.

Example 13. The operators $B_n\left(f(\sqrt{t}); x^2\right)$ were used in [17] in order to solve a problem raised in [21]. They fix e_0 and e_2, and have the Voronovskaya operator

$$\frac{1-x^2}{8}\left(\frac{d^2}{dx^2} - \frac{1}{x}\frac{d}{dx}\right).$$

Example 14. If in Section 4 we take $\gamma = \tau$, we get operators fixing τ and τ^2. Such operators were constructed also in [4].

Example 15. Operators fixing e^{ax} and e^{2ax}, $a > 0$, on $[0, \infty)$, with Voronovskaya operator

$$\frac{x}{2}\left(\frac{d^2}{dx^2} - 3a\frac{d}{dx} + 2a^2\right)$$

can be found in [2], see also Example 6.

Operators fixing the same functions, but on $[0, 1]$, were constructed in [14]. They have the Voronovskaya operator

$$\frac{x(1-x)}{2}\left(\frac{d^2}{dx^2} - 3a\frac{d}{dx} + 2a^2\right).$$

Remark 7. Other general Voronovskaya type formulas, related to operators fixing some functions, can be found in [1, 5, 6, 18, 25, 29, 30, 31, 32].

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