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Weighted Discrete Universality of the Riemann Zeta-Function

Antanas Laurinčikas[a], Darius Šiaučiūnas[b] and Gediminas Vadeikis[a]

[a] *Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University*
Naugarduko str. 24, LT-03225 Vilnius, Lithuania
[b] *Institute of Regional Development, Šiauliai University*
P. Višinskio str. 25, LT-76351 Šiauliai, Lithuania
E-mail(*corresp.*): darius.siauciunas@su.lt
E-mail: antanas.laurincikas@mif.vu.lt
E-mail: gediminas.vadeikis@mif.vu.lt

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Abstract. It is well known that the Riemann zeta-function is universal in the Voronin sense, i.e., its shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions. The universality of $\zeta(s)$ is called discrete if τ take values from a certain discrete set. In the paper, we obtain a weighted discrete universality theorem for $\zeta(s)$ when τ takes values from the arithmetic progression $\{kh : k \in \mathbb{N}\}$ with arbitrary fixed $h>0$. For this, two types of h are considered.

Keywords: approximation of analytic functions, Mergelyan theorem, Riemann zeta-function, universality, weak convergence.

AMS Subject Classification: 11M06; 41A30.

1 Introduction

The Riemann zeta-function $\zeta(s)$, $s=\sigma+it$,

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}, \quad \sigma>1,$$

since Riemann's and even Euler's times surprises mathematicians by the extensive field of applications and denseness of the set of its values. It is well known the role of $\zeta(s)$ in the theory of distribution of prime numbers and in other problems of arithmetic, however, we, in this paper, prefer the denseness properties of $\zeta(s)$.

In the second decade of the last century, H. Bohr discovered [4] that the function $\zeta(s)$ takes every non-zero value infinitely many times in the strip $\{s \in \mathbb{C} : 1 < \sigma < 1 + \delta\}$ with any $\delta > 0$. H. Bohr and R. Courant proved [5] that, for fixed σ, $\frac{1}{2} < \sigma \leqslant 1$, the set

$$\{\zeta(\sigma + it) : t \in \mathbb{R}\} \tag{1.1}$$

is dense in $\mathbb{C}$. S.M. Voronin significantly generalized the above results. He obtained [25] that the set

$$\{(\zeta(s_1 + i\tau), \dots, \zeta(s_n + i\tau)) : \tau \in \mathbb{R}\}$$

with any fixed numbers $s_1, \dots, s_n$, $\frac{1}{2} < \mathrm{Re} s_k < 1$, $1 \leqslant k \leqslant n$, and $s_k \neq s_m$ for $k \neq m$, and the set

$$\left\{\left(\zeta(s + i\tau), \zeta'(s + i\tau), \dots, \zeta^{(n-1)}(s + i\tau)\right) : \tau \in \mathbb{R}\right\}$$

with every fixed s, $\frac{1}{2} < \sigma < 1$, are dense in $\mathbb{C}^n$. However, a much more important merit of Voronin is his so-called universality theorem for the function $\zeta(s)$ [26]. This theorem asserts that a wide class of analytic functions can be approximated by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. For a modern version of the Voronin universality theorem, it is convenient to use the following notation. Let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$, $K \in \mathcal{K}$, the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then the following theorem is true.

Theorem 1. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Here meas$A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. By Theorem 1, the set of shifts $\zeta(s + i\tau)$ approximating a given function from $H_0(K)$ has a positive lower density, thus, it is infinite. Also, Theorem 1 can be considered as an infinite-dimensional version of the Bohr-Courant theorem on denseness of the set (1.1). The proof of Theorem 1 is given in [1] (in slightly different form), and in [9], [11], [24].

Theorem 1 is of continuous type: τ in $\zeta(s + i\tau)$ can take arbitrary real values. Also, a discrete version of Theorem 1 is known when τ takes values from a certain discrete set. Let $h > 0$ be a fixed number.

Theorem 2. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N} \# \left\{1 \leqslant k \leqslant N : \sup_{s \in K} |\zeta(s + ikh) - f(s)| < \varepsilon\right\} > 0.$$

Here $\#A$ denotes the cardinality of the set A. The proof of Theorem 2 can be found in [22] and [1]. For shifts $\zeta(s+ik^{\alpha}h)$ with fixed α, $0<\alpha<1$, Theorem 2 is given in [6]. In [15, 21] and [7, 8, 13, 16], more general shifts of Dirichlet L-functions and Riemann zeta-function, respectively, were considered. We note that discrete universality theorems for zeta-functions sometimes are more convenient for practical applications, an example of this is the paper [3].

In [10], a weighted version of Theorem 1 was proposed. Let $w(t)$ be a function of bounded variation on $[T_0,\infty)$ with some $T_0>0$ such that the variation $\mathrm{V}_a^b w$ on $[a,b]$ satisfies the inequality $\mathrm{V}_a^b w \leqslant cw(a)$ with a certain constant $c>0$ for any subinterval $[a,b]\subset[T_0,\infty)$. Let

$$U_T = U(T,w) = \int_{T_0}^{T} w(t)\mathrm{d}t,$$

and let $\lim_{T\to\infty} U(T,w)=+\infty$. Moreover, let I_A denote the indicator function of the set A. Then we have the following generalization of Theorem 1.

Theorem 3. *Suppose that the function $w(t)$ satisfies the above hypotheses. Let $K\in\mathcal{K}$ and $f(s)\in H_0(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{U_T}\int_{T_0}^{T} w(\tau) I_{\left\{\tau:\sup\limits_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\right\}}(\tau)\mathrm{d}\tau>0.$$

To be precise, in [10], Theorem 3 was proved under a certain additional hypothesis on the function $w(t)$ which is a weighted version of the classical Birkhoff-Khintchine ergodic theorem. In [18], this technical hypothesis was removed. A generalization of Theorem 3 for Matsumoto zeta-functions was given in [12]. In [17], a weighted discrete universality theorem with the sequence $\{k^{\alpha}h\}$, $0<\alpha<1$, for the periodic zeta-function was obtained.

The aim of this paper is a weighted discrete universality theorem for the Riemann zeta-function. Let $w(t)$ be a real non-negative function having a continuous derivative on $[\frac{1}{2},\infty)$ such that

$$\lim_{N\to\infty} V_N=+\infty,\quad V_N=\sum_{k=1}^{N} w(k),\qquad \int_1^N u|w'(u)|\mathrm{d}u \ll V_N,\quad N\to\infty.$$

Denote by W the class of functions $w(t)$ satisfying the above hypotheses. Suppose that h is a fixed positive number.

Theorem 4. *Suppose that $w(t)\in W$. Let $K\in\mathcal{K}$ and $f(s)\in H_0(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{N\to\infty}\frac{1}{V_N}\sum_{k=1}^{N} w(k) I_{\left\{k:\sup\limits_{s\in K}|\zeta(s+ikh)-f(s)|<\varepsilon\right\}}(k)>0.$$

For example, the function $w(t)=\frac{\sin(\log t)+1}{t}$ is not monotonically decreasing and $w(t)\in W$.

Theorem 4 has the following modification.

Theorem 5. *Suppose that $w(t) \in W$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I_{\left\{k:\sup\limits_{s\in K} |\zeta(s+ikh)-f(s)|<\varepsilon\right\}}(k) > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For proving of the above universality theorems, we will apply the probabilistic approach.

2 Limit theorems

We remind that $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$, and by $H(D)$ denote the space of analytic functions on D endowed with the topology of uniform convergence on compacta. The space $H(D)$ is metrisable. There exists a sequence of compact subsets $\{K_l : l \in \mathbb{N}\} \subset D$ such that $D = \bigcup\limits_{l=1}^{\infty} K_l$, $K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some $l \in \mathbb{N}$. For $g_1, g_2 \in H(D)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K} |g_1(s) - g_2(s)|}.$$

Then ρ is a metric on $H(D)$ which induces its topology of uniform convergence on compacta.

Denote by $\mathcal{B}(X)$ the Borel σ-field of the space X, and, for $A \in \mathcal{B}(H(D))$, define

$$P_N(A) = P_{N,w,h}(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I_{\{k:\zeta(s+ikh)\in A\}}(k).$$

In this section, we will consider the weak convergence of $P_{N,w,h}$ as $N \to \infty$. We say that $h > 0$ is of type 1 if $\exp\left\{\frac{2\pi m}{h}\right\}$ is an irrational number for all $m \in \mathbb{Z} \setminus \{0\}$, and $h > 0$ is of type 2 if h is not of type 1. We will examine separately the cases of types 1 and 2.

As usual, we start with one topological structure. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ and $\Omega = \prod_p \gamma_p$, where $\gamma_p = \gamma$ for all primes p. By the Tikhonov theorem, the torus Ω with the product topology and pointwise multiplication is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and this gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\mathbb{P}$ be the set of all prime numbers, and let $\omega(p)$ denote the projection of $\omega \in \Omega$ to the circle γ_p, $p \in \mathbb{P}$. For $A \in \mathcal{B}(\Omega)$, define

$$Q_N(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I_{\{k:(p^{-ikh}:p\in\mathbb{P})\in A\}}(k).$$

Lemma 1. *Suppose that $w(t) \in W$ and h is of type 1. Then Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. We apply the Fourier transform method. Let $g_N(\underline{k})$, $\underline{k} = (k_p : k_p \in \mathbb{Z},\ p \in \mathbb{P})$, be the Fourier transform of Q_N. Then we have that

$$g_N(\underline{k}) = \int_\Omega \prod_p{}^* \omega^{k_p}(p) \mathrm{d}Q_N,$$

where the sign "$*$" means that only a finite number of integers k_p are distinct from zero. Thus, by the definition of Q_N,

$$g_N(\underline{k}) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) \prod_p{}^* p^{-ikk_p h} = \frac{1}{V_N} \sum_{k=1}^{N} w(k) \exp\Big\{ - ikh \sum_p{}^* k_p \log p \Big\}. \tag{2.1}$$

Obviously,

$$g_N(\underline{0}) = 1. \tag{2.2}$$

If $\underline{k} \neq \underline{0}$, then

$$\sum_p{}^* k_p \log p \neq 0,$$

since the logarithms of prime numbers are linearly independent over the field of rational numbers. Thus,

$$\exp\left\{ -ih \sum_p{}^* k_p \log p \right\} \neq 1. \tag{2.3}$$

Indeed, if inequality (2.3) is not true, then

$$\sum_p{}^* k_p \log p = \frac{2\pi r}{h}, \quad \prod_p{}^* p^{-k_p} = \exp\left\{ \frac{2\pi r}{h} \right\}$$

with some $r \in \mathbb{Z} \setminus \{0\}$. However, the left-hand side of this equality is a rational number, and we arrive to the contradiction that h is of type 1. Thus, (2.3) is true, and we find that, for $u \geqslant 1$,

$$\sum_{k \leqslant u} \exp\left\{ -ikh \sum_p{}^* k_p \log p \right\}$$
$$= \frac{\exp\left\{ -ih \sum_p^* k_p \log p \right\} - \exp\left\{ i([u]+1)h \sum_p^* k_p \log p \right\}}{1 - \exp\left\{ -ih \sum_p^* k_p \log p \right\}} \overset{def}{=} \varSigma(u).$$

Hence, in view of (2.1), for $\underline{k} \neq \underline{0}$,

$$g_N(\underline{k}) = \frac{w(N)\varSigma(N)}{V_N} - \frac{1}{V_N} \int_1^N \varSigma(u) w'(u) \mathrm{d}u.$$

Since the function $\varSigma(u)$ is bounded by a constant not depending of u, we find that, for $\underline{k} \neq \underline{0}$,

$$\lim_{N \to \infty} g_N(\underline{k}) = 0.$$

This together with (2.2) proves the lemma. □

Lemma 1 implies a weighted discrete universality theorem for absolutely convergent Dirichlet series. Let $\theta > \frac{1}{2}$ be a fixed number, and

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\theta}\right\}, \quad m, n \in \mathbb{N},$$

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s}, \qquad \zeta_n(s,\omega) = \sum_{m=1}^{\infty} \frac{\omega(m) v_n(m)}{m^s},$$

where

$$\omega(m) = \prod_{\substack{p^\alpha | m \\ p^{\alpha+1} \nmid m}} \omega^\alpha(p), \quad m \in \mathbb{N}.$$

Then the series for $\zeta_n(s)$ and $\zeta_n(s,\omega)$ are absolutely convergent for $\sigma > \frac{1}{2}$ [11]. From this, it follows that the function $u_n : \Omega \to H(D)$, $u_n(\omega) = \zeta_n(s,\omega)$, is continuous. Let $R_n = m_H u_n^{-1}$, where

$$R_n(A) = m_H u_n^{-1}(A) = m_H(u_n^{-1} A), \quad A \in \mathcal{B}(H(D)).$$

Moreover, let

$$P_{N,n}(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I_{\{k:\zeta_n(s+ikh)\in A\}}(k), \quad A \in \mathcal{B}(H(D)).$$

It is not difficult to see that $P_{N,n} = Q_N u_n^{-1}$. This, the continuity of u_n and Lemma 1 lead to

Lemma 2. *Suppose that $w(t) \in W$ and h is of type 1. Then $P_{N,n}$ converges weakly to R_n as $N \to \infty$.*

The weak convergence of $P_{N,n}$ is a starting point for proving the weak convergence for P_N as $N \to \infty$. The investigation of P_N also requires an approximation of $\zeta(s)$ by $\zeta_n(s)$. Let $l_n(s) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) n^s$, where $\Gamma(s)$ is the Euler gamma-function. Then [11], for $\sigma > \frac{1}{2}$, the integral representation

$$\zeta_n(s) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s+z) l_n(z) \frac{\mathrm{d}z}{z} \tag{2.4}$$

is true. Using the well-known estimates

$$\int_{1/2}^{T} |\zeta(\sigma + it)|^2 \mathrm{d}t \ll T, \quad \int_{1/2}^{T} |\zeta'(\sigma + it)|^2 \mathrm{d}t \ll T,$$

we find that, for $\frac{1}{2} < \sigma < 1$ and $\tau \in \mathbb{R}$,

$$\int_{1/2}^{T} |\zeta(\sigma + it + i\tau)|^2 \,\mathrm{d}t \ll T(1 + |\tau|)$$

and

$$\int_{1/2}^{T} \left|\zeta'(\sigma+it+i\tau)\right|^2 \mathrm{d}t \ll T(1+|\tau|).$$

These estimates together with Gallagher lemma, see, for example, [20, Lemma 1.4], give, for $\frac{1}{2}<\sigma<1$ and $\tau\in\mathbb{R}$, the bound

$$\sum_{k=1}^{N} |\zeta(\sigma+ikh+i\tau)|^2 \ll \int_{1/2}^{(N+1/2)h} |\zeta(\sigma+it+i\tau)|^2 \mathrm{d}t$$
$$+\left(\int_{1/2}^{(N+1/2)h} |\zeta(\sigma+it+i\tau|^2 \mathrm{d}t \int_{1/2}^{(N+1/2)h} |\zeta'(\sigma+it+i\tau)|^2 \mathrm{d}t\right)^{1/2} \ll N(1+|\tau|).$$

Hence, for the same σ and τ,

$$\sum_{k=1}^{N} w(k)|\zeta(\sigma+ikh+i\tau)|^2 \ll w(N)\sum_{k=1}^{N} |\zeta(\sigma+ikh+i\tau)|^2 + (1+|\tau|)$$
$$\times \int_{1}^{N} u|w'(u)|\mathrm{d}u \ll Nw(N)(1+|\tau|) + V_N(1+|\tau|) \ll V_N(1+|\tau|), \quad (2.5)$$

because

$$Nw(N)=\sum_{k=1}^{N} w(k) + \int_{1}^{N}\left(\sum_{k\leqslant u} 1\right) w'(u)\mathrm{d}u \ll V_N.$$

Let $K\subset D$ be a compact set. Then (2.4), (2.5), the residue theorem and Cauchy integral formula imply the equality

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{V_N}\sum_{k=1}^{N} w(k)\sup_{s\in K}|\zeta(s+ikh)-\zeta_n(s+ikh)|=0. \quad (2.6)$$

Now, (2.6) together with the definition of the metric ρ yields the following lemma.

Lemma 3. *Suppose that $w(t)\in W$. Then the equality*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{V_N}\sum_{k=1}^{N} w(k)\rho\left(\zeta(s+ikh),\zeta_n(s+ikh)\right)=0$$

is true for every fixed $h>0$.

Now, we are in position to prove a weighted discrete limit theorem for the function $\zeta(s)$. On the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$, define the $H(D)$-valued random element $\zeta(s,\omega)$ by the Euler product

$$\zeta(s,\omega)=\prod_p\left(1-\frac{\omega(p)}{p^s}\right)^{-1}.$$

The latter product, for almost all $\omega \in \Omega$, is uniformly convergent on compact subsets of the strip D [11]. Denote by P_ζ the distribution of the random element $\zeta(s,\omega)$, i.e.,

$$P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s,\omega) \in A), \quad A \in \mathcal{B}(H(D)).$$

Theorem 6. *Suppose that $w(t) \in W$ and $h > 0$ is of the type 1. Then P_N converges weakly to P_ζ as $N \to \infty$. Moreover, the support of P_ζ is the set $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.*

Proof. We will prove that R_n, as $n \to \infty$, converges weakly to a certain probability measure P, and that P_N, as $N \to \infty$, also converges weakly to P.

Let θ_N be a random variable defined on a certain probability space with probability measure μ and having the distribution

$$\mu(\theta_N = kh) = \frac{w(k)}{V_N}, \quad k = 1, \dots, N.$$

Moreover, let $Y_{N,n} = Y_{N,n}(s)$ be an $H(D)$-valued random element defined by

$$Y_{N,n}(s) = \zeta_n(s + i\theta_N),$$

and let $Y_n = Y_n(s)$ be an $H(D)$-valued random element with the distribution R_n. Then, by Lemma 2,

$$Y_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} Y_n. \tag{2.7}$$

Using the absolute convergence of the series for $\zeta_n(s)$, it can be proved by a method of [11] that the family of probability measures $\{R_n : n \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H(D)$ such that

$$R_n(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Hence, by the Prokhorov theorem [2], this family is relatively compact. Therefore, each sequence of $\{R_n\}$ contains a subsequence $\{R_{n_r}\}$ weakly convergent, as $r \to \infty$, to a certain probability measure P on $(H(D), \mathcal{B}(H(D)))$. In other words,

$$Y_{n_r} \xrightarrow[r\to\infty]{\mathcal{D}} P. \tag{2.8}$$

Define one more $H(D)$-valued random element

$$X_N = X_N(s) = \zeta(s + i\theta_N).$$

Then the application of Lemma 3 gives, for $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} \mu\left(\rho(X_N(s), Y_{N,n}(s)) \geqslant \varepsilon\right) \\
&\quad = \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I_{\{k:\rho(\zeta(s+ikh),\zeta_n(s+ikh))\geqslant\varepsilon\}}(k) \\
&\quad \leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon V_N} \sum_{k=1}^{N} w(k)\rho(\zeta(s+ikh), \zeta_n(s+ikh)) = 0.
\end{aligned}$$

This equality, (2.7) and (2.8) show that all hypotheses of Theorem 4.2 of [2] are satisfied, therefore,

$$X_N \xrightarrow[N\to\infty]{\mathcal{D}} P, \tag{2.9}$$

or P_N converges weakly to P as $N \to \infty$. Moreover, in virtue of (2.9), the measure P is independent of the sequence Y_{n_r}. Since the family $\{R_n\}$ is relatively compact, from this, we obtain that R_n converges weakly to P as $n \to \infty$. Thus, P_N, as $N \to \infty$, converges weakly to the limit measure P of R_n as $n \to \infty$. However, by the proof of a limit theorem for

$$\frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \zeta(s+i\tau) \in A\right\}, \quad A \in \mathcal{B}(H(D)),$$

it is known [11] that R_n, as $n \to \infty$, converges weakly to P_ζ, and the support of P_ζ is the set S. Therefore, the same statement is also true for P_N, and the theorem is proved. □

The case of h of type 2 is a more complicated. We must construct a new probability space different from $(\Omega, \mathcal{B}(\Omega), m_H)$. We will index by h the notation related to h of type 2.

Now suppose that $h > 0$ is of type 2. Then there exists the smallest $m_0 \in \mathbb{N}$ such the number $\exp\left\{\frac{2\pi m_0}{h}\right\}$ is rational. We put

$$\exp\left\{\frac{2\pi m_0}{h}\right\} = \frac{a}{b}, \quad a, b \in \mathbb{N},\ (a,b) = 1.$$

Define the set

$$\mathbb{P}_0 = \left\{p \in \mathbb{P} : \frac{a}{b} = \prod_{p\in\mathbb{P}} p^{\alpha_p} \text{ with } \alpha_p \neq 0\right\}.$$

Denote by Ω_h the closed subgroup of Ω generated by the element $\left\{p^{-ih} : p{\in}\mathbb{P}\right\}$. By Lemma 1 of [14], if h is of type 2, then

$$\Omega_h = \{\omega \in \Omega : \omega(a) = \omega(b)\}.$$

On $(\Omega_h, \mathcal{B}(\Omega_h))$, the probability Haar measure m_H^h exists, and we obtain the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$. By (3.1) of [14], we have that the characters χ of the group Ω_h are of the form

$$\chi(\omega) = {\prod_{p\in\mathbb{P}\setminus\mathbb{P}_0}}^{*} \omega^{k_p}(p) \prod_{p\in\mathbb{P}_0} \omega^{k_p+l\alpha_p}(p), \quad l \in \mathbb{Z}. \tag{2.10}$$

Now, we are ready to prove an analogue of Lemma 1 for h of type 2. For $A \in \mathcal{B}(\Omega_h)$, define

$$Q_{N,h}(A) = \frac{1}{V_N}\sum_{k=1}^{N} w(k) I_{\{k:(p^{-ikh}:p\in\mathbb{P})\in A\}}(k).$$

Lemma 4. *Suppose that h is of type 2. Then $Q_{N,h}$ converges weakly the Haar measure m_H^h as $N \to \infty$.*

Proof. In view of (2.10), we have that the Fourier transform $g_{N,h}(\underline{k})$, $\underline{k} = (k_p : k_p \in \mathbb{Z},\ p \in \mathbb{P})$, of $Q_{N,h}$ is of the form

$$g_{N,h}(\underline{k}) = \int_{\Omega_h} \chi(\omega)\mathrm{d}Q_{N,h} = \frac{1}{V_N}\sum_{k=1}^{N} w(k) \prod_{p\in\mathbb{P}\setminus\mathbb{P}_0}^{*} p^{-ikk_ph} \prod_{p\in\mathbb{P}_0} p^{-ikh(k_p+l\alpha_p)}, \quad l \in \mathbb{Z}. \tag{2.11}$$

If $k_p = 0$ for all $p \in \mathbb{P}\setminus\mathbb{P}_0$ and $k_p = r\alpha_p$ for all $p \in \mathbb{P}_0$ with some $r \in \mathbb{Z}$ (case 1), then

$$g_{N,h}(\underline{k}) = 1, \tag{2.12}$$

because $\prod_{p\in\mathbb{P}_0} \omega^{d\alpha_p}(p) = 1$ with $d \in \mathbb{Z}$.

Now, suppose that $k_p \neq 0$ for some $p \in \mathbb{P}\setminus\mathbb{P}_0$, or there does not exist $r \in \mathbb{Z}$ such that $k_p = r\alpha_p$ for all $p \in \mathbb{P}_0$ (case 2). In [14], it was obtained that

$$\exp\{-ihA_p(k_p, l\alpha_p)\} \neq 1,$$

where

$$A_p(k_p, l\alpha_p) = \sum_{p\in\mathbb{P}\setminus\mathbb{P}_0}^{*} k_p \log p + \sum_{p\in\mathbb{P}_0} (k_p + l\alpha_p)\log p, \quad l \in \mathbb{Z}.$$

Hence, we find that, for $u \geqslant 1$,

$$\sum_{k\leqslant u} \exp\{-ikhA_p(k_p, l\alpha_p)\} = \frac{\exp\{-ihA_p(k_p, l\alpha_p)\} - \exp\{-ih([u]+1)A_p(k_p, l\alpha_p)\}}{1 - \exp\{-ihA_p(k_p, l\alpha_p)\}} \stackrel{def}{=} \Sigma_h(u).$$

Therefore, in view of (2.11),

$$g_{N,h}(\underline{k}) = \frac{w(N)\Sigma_h(N)}{V_N} - \frac{1}{V_N}\int_1^N \Sigma_h(u)w'(u)\mathrm{d}u.$$

Using the properties of the function w, hence we find that

$$g_{N,h}(\underline{k}) = 0.$$

This together with (2.12) shows that

$$\lim_{N\to\infty} g_{N,h}(\underline{k}) = \begin{cases} 1, & \text{in the case 1,} \\ 0, & \text{in the case 2.} \end{cases}$$

Since the right-hand side of the equality is the Fourier transform of the Haar measure m_H^h, the lemma follows by a continuity theorem for probability measures on compact groups. □

Now, together with $P_{N,n,h}$, consider

$$\hat{P}_{N,n,h}(A) = \frac{1}{V_N}\sum_{k=1}^{N} w(k) I_{\{k:\zeta_{n,h}(s+ikh,\omega)\in A\}}(k), \quad A \in \mathcal{B}(H(D)),$$

with $\omega \in \Omega_h$.

Lemma 5. *Suppose that $w(t) \in W$ and h is of type 2. Then $P_{N,n,h}$ and $\hat{P}_{N,n,h}$ both converge weakly to the measure $m_H^h u_{n,h}^{-1}$ as $N \to \infty$, where $u_{n,h} : \Omega_h \to H(D)$ is given by $u_{n,h}(\omega) = \zeta_{n,h}(s, \omega)$, $\omega \in \Omega_h$.*

Proof. By proving Lemma 2, in view of Lemma 4, we have that $P_{N,n,h}$ converges weakly to $m_H^h u_{n,h}^{-1}$ as $N \to \infty$. Similarly, we obtain that if $\hat{u}_{n,h}(\hat{\omega}) : \Omega_h \to H(D)$ is given by

$$\hat{u}_{n,h}(\hat{\omega}) = \zeta_n(s, \omega\hat{\omega}), \quad \hat{\omega} \in \Omega_h,$$

then $\hat{P}_{N,n,h}$ converges weakly to $m_H^h \hat{u}_{n,h}^{-1}$. However, $\hat{u}_{n,h} = u_{n,h}(u)$, where $u : \Omega_h \to \Omega_h$ is given by $u(\hat{\omega}) = \omega\hat{\omega}$. This and the invariance of the Haar measure m_H^h show that $m_H^h \hat{u}_{n,h}^{-1} = m_H^h u_{n,h}^{-1}$. □

For further considerations, we need some elements of the ergodic theory. Let $a_h = (p^{-ih} : p \in \mathbb{P})$. Then a_h is an element of Ω_h. Define the transformation $\varphi_h(\omega)$ of Ω_h by

$$\varphi_h(\omega) = a_h \omega, \quad \omega \in \Omega_h.$$

Then we have that φ_h is a measurable measure preserving transformation on the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$. We recall that a set $A \in \mathcal{B}(\Omega_h)$ is called invariant with respect to φ_h if the sets A and $\varphi_h(A)$ can differ from each other at most by a set of m_H^h-measure zero. The transformation φ_h is called ergodic if the σ-field of invariant sets of Ω_h consists only of the sets having m_H^h-measure 1 or 0.

Lemma 6. *Suppose that h is of type 2. Then the transformation φ_h is ergodic.*

Proof of the lemma is given in [14, Lemma 3].

Let, for $\omega \in \Omega_h$,

$$\zeta_h(s, \omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

The first application of Lemma 6 is devoted to the discrete mean square of $\zeta_h(s, \omega)$.

Lemma 7. *Suppose that $w(t) \in W$, $h > 0$ is of type 2, σ, $\frac{1}{2} < \sigma < 1$, is fixed and $t \in \mathbb{R}$. Then, for almost all $\omega \in \Omega_h$,*

$$\sum_{k=1}^{N} w(k) |\zeta_h(\sigma + it + ikh, \omega)|^2 \ll V_N(1 + |t|).$$

Proof. We have that $\zeta_h(s, \omega)$ coincides with the restriction of the random element $\zeta(s, \omega)$ to the space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$. First we consider the expectation $\mathbb{E}|\zeta_h(\sigma + it, \omega)|^2$. We write $\zeta_h(s, \omega)$ in the form

$$\zeta_h(\sigma + it, \omega) = \prod_{p \in \mathbb{P}_0} \left(1 - \frac{\omega(p)}{p^{\sigma + it}}\right)^{-1} \prod_{p \in \mathbb{P} \setminus \mathbb{P}_0} \left(1 - \frac{\omega(p)}{p^{\sigma + it}}\right)^{-1} \stackrel{def}{=} X_1 X_2. \quad (2.13)$$

The random elements X_1 and X_2 are independent, moreover, for almost all $\omega \in \Omega_h$,

$$X_2 = \sum_m{}' \frac{\omega(m)}{m^{\sigma+it}},$$

where the sign $'$ means that the summing runs over $m = 1$ and $m \in \mathbb{N}$ with the canonical representation consisting only of primes $p \in \mathbb{P} \setminus \mathbb{P}_0$. In the series for X_2, the random variables are orthogonal, therefore,

$$\mathbb{E}|X_2|^2 = \sum_m{}' \frac{1}{m^{2\sigma}} < \infty.$$

Clearly, $\mathbb{E}|X_1|^2$ is bounded by a constant. Therefore, there exists a finite constant $c > 0$ such that, for $\frac{1}{2} < \sigma < 1$ and $t \in \mathbb{R}$,

$$\mathbb{E}|\zeta_h(\sigma+it,\omega)|^2 = \mathbb{E}|X_1|^2 \mathbb{E}|X_2|^2 \leqslant c.$$

Then (2.13), Lemma 6, the Birkhoff-Khintchine ergodic theorem, see, for example, [23], and the definition of the transformation φ_h show that, for $\frac{1}{2} < \sigma < 1$ and $|t_0| \leqslant h$,

$$\begin{aligned} \sum_{k=1}^N |\zeta_h(\sigma+it_0+ikh,\omega)|^2 &= \sum_{k=1}^N |\zeta_h(\sigma+it_0,\varphi_h^k(\omega))|^2 \\ &= N\mathbb{E}|\zeta_h(\sigma+it_0,\omega)|^2(1+o(1)) \ll N \end{aligned}$$

for almost all $\omega \in \Omega_h$ as $N \to \infty$. Hence, denoting by $[u]$ the integer part of $u \in \mathbb{R}$, for $\frac{1}{2} < \sigma < 1$ and $t \in \mathbb{R}$, we find that

$$\sum_{k=1}^N |\zeta_h(\sigma+it+ikh,\omega)|^2 = \sum_{k=1+[t/h]}^{N+[t/h]} |\zeta_h(\sigma+it_0+ikh,\omega)|^2 \ll N(1+|t|)$$

for almost all $\omega \in \Omega_h$. From this, summing by parts, we obtain the estimate of the lemma. □

Similarly to the proof of Lemma 3, we arrive, by using Lemma 7, to

Lemma 8. *Suppose that $w(t) \in W$ and $h > 0$ is of type 2. Then, for almost all $\omega \in \Omega_h$,*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^N w(k) \rho\left(\zeta_h(s+ikh,\omega), \zeta_{n,h}(s+ikh,\omega)\right) = 0.$$

For $\omega \in \Omega_h$, additionally to the measure $P_{N,h}$, define

$$\hat{P}_{N,h}(A) = \frac{1}{V_N} \sum_{k=1}^N w(k) I_{\{k:\zeta_h(s+ikh,\omega)\in A\}}(k), \quad A \in \mathcal{B}(H(D)).$$

Then, using Lemmas 3, 5 and 8, and repeating the first part of the proof of Theorem 6, we obtain

Lemma 9. *Suppose that $w(t) \in W$ and $h > 0$ is of type 2. Then, on $(H(D), \mathcal{B}(H(D)))$, there exists a probability measure P_h such that $P_{N,h}$ and $\hat{P}_{N,h}$ both converges weakly to P_h as $N \to \infty$.*

Denote by $P_{\zeta,h}$ the distribution of the random element $\zeta_h(s,\omega)$, $\omega \in \Omega_h$. Then we have the following analogue of Theorem 6.

Theorem 7. *Suppose that $w(t) \in W$ and $h > 0$ is of type 2. Then $P_{N,h}$ converges weakly $P_{\zeta,h}$ as $N \to \infty$. Moreover, the support of the measure $P_{\zeta,h}$ is the set S.*

Proof. In virtue of Lemma 9, it suffices to identify the measure P in that lemma, and to find the support of the limit measure. For the first problem, we will apply Lemma 6, and the Birkhoff-Khintchine theorem. Let A be a continuity set of P. On the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$, define the random variable ξ by the formula

$$\xi(\omega) = \begin{cases} 1, & \text{if} \quad \zeta_h(s,\omega) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have that

$$\mathbb{E}\xi = \int_{\Omega_h} \xi(\omega) \mathrm{d}m_H^h = P_{\zeta,h}(A). \tag{2.14}$$

Moreover, by Lemma 9,

$$\lim_{N\to\infty} \hat{P}_N(A) = P_h(A). \tag{2.15}$$

In view of Lemma 6 and the Birkhoff-Khintchine theorem, for almost all $\omega \in \Omega_h$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \xi(\varphi_h^k(\omega)) = \mathbb{E}\xi.$$

Since $w \in W$, from this it follows that, for almost all $\omega \in \Omega_h$,

$$\lim_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k)\xi(\varphi_h^k(\omega)) = \mathbb{E}\xi. \tag{2.16}$$

However, by the definition of φ_h,

$$\frac{1}{V_N} \sum_{k=1}^{N} w(k)\xi(\varphi_h^k(\omega)) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I_{\{k:\zeta_h(s+ikh,\omega)\in A\}}(k) = \hat{P}_{N,h}(A).$$

Therefore, by (2.14) and (2.16),

$$\lim_{N\to\infty} \hat{P}_{N,h}(A) = P_{\zeta,h}(A).$$

This and (2.15) show that $P_h = P_{\zeta,h}$.

For finding the support of $P_{\zeta,h}$, we use the representation (2.13). For $p \in \mathbb{P} \setminus \mathbb{P}_0$, the random variables $\omega(p)$ are independent. Thus, by the proof of Lemma 6.5.5 from [11], we find that the support of the random element X_2 is the set S. Since the random elements X_1 and X_2 are independent and X_1 is not degenerate at zero, we obtain that the support of X_1X_2 is the set S, i.e., the support of the measure $P_{\zeta,h}$ is the set S. The theorem is proved. □

3 Proof of universality theorems

Theorems 4 and 5 follow from the limit theorems (Theorems 6 and 7), for $\zeta(s)$ as well as from the Mergelyan theorem [19] on the approximation of analytic functions by polynomials.

Proof. (Of Theorem 4). By the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K}\left|f(s)-e^{p(s)}\right|<\frac{\varepsilon}{2}. \tag{3.1}$$

For brevity, denote the limit measure in Theorems 6 and 7 by $\hat{P}_\zeta$, i.e.,

$$\hat{P}_\zeta=\begin{cases} P_\zeta, & \text{if } h \text{ is of type 1},\\ P_{\zeta,h}, & \text{if } h \text{ is of type 2},\end{cases} \qquad \hat{P}_N=\begin{cases} P_N, & \text{if } h \text{ is of type 1},\\ P_{N,h}, & \text{if } h \text{ is of type 2}.\end{cases}$$

Then we have that $\hat{P}_N$ converges weakly to $\hat{P}_\zeta$ as $N\to\infty$. Define the set

$$G_\varepsilon=\left\{g\in H(D): \sup_{s\in K}\left|g(s)-e^{p(s)}\right|<\frac{\varepsilon}{2}\right\}.$$

Since $e^{p(s)}\neq 0$, and, in view of Theorems 6 and 7, the support of the measure $\hat{P}_\zeta$ is the set S, the set G_ε is an open neighbourhood of an element of the support, therefore,

$$\hat{P}_\zeta(G_\varepsilon)>0. \tag{3.2}$$

Moreover, by the first parts of Theorems 6 and 7, and the equivalent of weak convergence of probability measures in terms of open sets [2, Theorem 2.1], we have that

$$\liminf_{N\to\infty}\hat{P}_N(G_\varepsilon)\geqslant\hat{P}_\zeta(G_\varepsilon).$$

This, (3.2) and the definitions of $\hat{P}_N$ and G_ε show that

$$\liminf_{N\to\infty}\frac{1}{V_N}\sum_{k=1}^{N}w(k)I_{\left\{k:\sup\limits_{s\in K}\left|\zeta(s+ikh)-e^{p(s)}\right|<\frac{\varepsilon}{2}\right\}}(k)>0. \tag{3.3}$$

It remains to replace $e^{p(s)}$ by $f(s)$ in the latter inequality. Suppose that k satisfies the inequality

$$\sup_{s\in K}\left|\zeta(s+ikh)-e^{p(s)}\right|<\frac{\varepsilon}{2}.$$

Then, in virtue of (3.1), the same k satisfies the inequality

$$\sup_{s\in K}\left|\zeta(s+ikh)-f(s)\right|<\varepsilon.$$

Therefore,

$$\left\{k:\sup_{s\in K}\left|\zeta(s+ikh)-e^{p(s)}\right|<\frac{\varepsilon}{2}\right\}\subset\left\{k:\sup_{s\in K}\left|\zeta(s+ikh)-f(s)\right|<\varepsilon\right\}.$$

This inclusion together with (3.3) proves the theorem. □

Proof. (Of Theorem 5). Define the set

$$\hat{G}_\varepsilon = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \Big\}.$$

Then $\partial \hat{G}_\varepsilon = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \Big\}$ is the boundary of $\hat{G}_\varepsilon$. Hence, $\partial \hat{G}_{\varepsilon_1} \cap \partial \hat{G}_{\varepsilon_2} = \varnothing$ if $\varepsilon_1 \neq \varepsilon_2$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$. Therefore, the set $\partial \hat{G}_\varepsilon$ can have a positive $\hat{P}_\zeta$-measure for at most countably many $\varepsilon > 0$. This means that the set $\hat{G}_\varepsilon$ is a continuity set of the measure $\hat{P}_\zeta$ for all but at most countably many $\varepsilon > 0$. Using Theorems 6 and 7, and the equivalent of weak convergence of probability measures in terms of continuity sets [2, Theorem 2.1], we have that

$$\lim_{N \to \infty} \hat{P}_N(\hat{G}_\varepsilon) = \hat{P}_\zeta(\hat{G}_\varepsilon) \tag{3.4}$$

for all but at most countably many $\varepsilon > 0$. Moreover, (3.1) shows that $G_\varepsilon \subset \hat{G}_\varepsilon$. Therefore, by (3.2), $\hat{P}_\zeta(\hat{G}_\varepsilon) > 0$. This, (3.4) and the definition of the set $\hat{G}_\varepsilon$ prove the theorem. □

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