A TOOL FOR MODAL ANALYSIS OF LAMINATED BENDING PLATES

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Abstract. A new finite element for modelling laminated bending plates was defined based on the effective triangular finite element of the discrete Kirchhoff's theory. The plates can be made of layers arranged in any order and consisting of different but orthotropic materials. The suggested finite element has 6 degrees of freedom in every node, i e 3 linear displacements and 3 rotations about the axis of coordinates. A mathematical model of the element describes stress and strain effects both in the plane of the element or perpendicular to it, except for shear. The suggested element can be used for calculating laminated plates or beams, not subjected to heavy shear stresses. Some numerical case studies are provided, while the results obtained are compared with the well-known analytical and numerical solutions.

Keywords: laminates, composites, composite structures, layered plates, linear analysis, non-linear analysis, finite element.

1. Introduction

Modern production technologies are used for manufacturing various composite materials. Composites, due to their outstanding mechanical properties, relatively low weight and a possibility to predetermine their characteristics, are widely used not only in high-tech areas, but in civil engineering as well. According to their structural and design characteristics, composite materials can be subdivided into reinforced materials shaped in various metal or non-metal moulds and laminated or layered structures obtained by combining layers of various materials. Therefore, the development of structures made of composite materials largely depends on the ability to model them.

It is hardly possible to review modelling problems associated with all available composite materials, therefore, the present paper addresses only the problems of modelling laminated bending plates. Though laminated structures are widely used, the theories of non-linear deformation and failure of modern layered composites as well as methods of mathematical modelling have not been fully developed and presented in detail yet. Because of anisotropic nature of laminated structures, all tension-compression and bending effects as well as the interaction of bending-membrane and membrane-shear effects should be considered [1–4]. Therefore, only a few assumptions simplifying the stress-strain state for the problems associated with the analysis of thin-wall bending plates can be applied.

A wide variety of finite elements are available for the analysis of structures made of commonly used materials. However, the problems associated with laminated twin-wall bending plates require a special kind of layered finite elements. Precise finite elements [5–9] are often hardly realisable in application programs due to the complexity of a mathematical model. Therefore, the need for sufficiently accurate, efficient and applicable finite elements to be used in laminated bending plates, which do not require any cross-section symmetry, still remains.

In the present paper, new types of matrix expressions allowing for evaluation of all above-mentioned membrane and bending effects are offered for the laminated anisotropic triangular finite element DKT_CST [10]. This finite element can be used for modelling anisotropic laminated bending plates consisting of layers of orthotropic material arranged in any order, as well as the particular zones of similar plates or beams, when shear deformations are insignificant.

2. Mathematical modelling of problems

A finite element may be considered to be completely defined, when it is applied to solve linear and non-linear static problems as well as modal analysis [11] and its accuracy is determined, because all characteristic structural matrices of the element should be generated for these problems. Below mathematical models of the considered problems are provided.

The following equation applies to a linear system of finite elements:

$$[K_0]\boldsymbol{\delta} = \mathbf{F}, \qquad (1)$$

where $[K_0]$ is a linear stiffness matrix; $\boldsymbol{\delta}$ – a displacement vector; $\mathbf{F}$ – a vector of loads applied to the system.

Due to membrane stresses, actual plate displacements are much smaller than those determined by the theory of linearity. A discrete problem model may be expressed by a system of non-linear algebraic equations:

$$[K(\boldsymbol{\delta})]\boldsymbol{\delta} - \mathbf{F} = 0 , \qquad (2)$$

where the stiffness matrix depends on displacements. The iterative Newton-Raphson method [12, 13] was used for solving a system of non-linear equations. According to this approach, a rough solution $\boldsymbol{\delta}_n$, where a connection error of external and internal forces $\boldsymbol{\psi}_n \neq 0$, was refined by solving a system of linear equations at every iteration:

$$\Delta\boldsymbol{\delta}_{n+1} = -[K_T]_n^{-1}\boldsymbol{\psi}_n , \qquad (3)$$

$$[K_T] = [K_0] + [K_\sigma] + [K_L] , \qquad (4)$$

where $[K_T]$ is a tangent stiffness matrix; $[K_\sigma]$ – a matrix of initial stresses; $[K_L]$ – a matrix of large displacements. A connection error $\boldsymbol{\psi}_n$ is calculated by the stresses $\boldsymbol{\sigma}_n$:

$$\boldsymbol{\psi}(\boldsymbol{\delta}) = \int_V [\overline{B}]^T \boldsymbol{\sigma} dV - \mathbf{F} = 0 , \qquad (5)$$

where $[\overline{B}]$ is a non-linear matrix relating deformations to displacements. A particular case of a geometrically non-linear problem is the problem of initial stability [11], when matrix $[K_L] = 0$.

The dynamic problem is expressed by the equation:

$$[K_0]\boldsymbol{\delta} + [C]\frac{\partial}{\partial t}\boldsymbol{\delta} + [M]\frac{\partial^2}{\partial t^2}\boldsymbol{\delta} + \mathbf{F} = 0 , \qquad (6)$$

where $[C]$ and $[M]$ are damping and mass matrices. A particular dynamic problem is an eigenvalue problem [11, 14], when matrix $[C] = 0$ and $\mathbf{F} = 0$. This problem is expressed by the equation (6) of the form:

$$[K_0]\boldsymbol{\delta} = \lambda[M]\boldsymbol{\delta} , \qquad (7)$$

where λ denotes natural frequencies.

3. Definition of the finite element

The stress-strain relation for laminated plates [15, 16] is expressed in the following way:

$$\mathbf{N} = \lfloor D^{pl} \rfloor \mathbf{e}^0 + \lfloor D^{plb} \rfloor \mathbf{k} ,$$

$$\mathbf{M} = \lfloor D^{plb} \rfloor \mathbf{e}^0 + \lfloor D^{b} \rfloor \mathbf{k} , \qquad (8)$$

where $\mathbf{N}$ and $\mathbf{M}$ are membrane and bending stresses; $\mathbf{e}^0$ denotes midsurface membrane strains; $\mathbf{k}$ – curve vector; $\lfloor D^{pl} \rfloor, \lfloor D^{plb} \rfloor, \lfloor D^{b} \rfloor$ – accumulative constitutive matrices obtained by combining constitutive matrices of layers [17–19]:

$$[D^{pl}] = \sum_{k=1}^{n}[D_k](z_k - z_{k-1}),$$

$$[D^{plb}] = \frac{1}{2}\sum_{k=1}^{n}[D_k](z_k^2 - z_{k-1}^2), \qquad (9)$$

$$[D^{b}] = \frac{1}{3}\sum_{k=1}^{n}[D_k](z_k^3 - z_{k-1}^3),$$

where k is the layer's number; $[D_k]$ – the k-th layer constitutive matrix obtained by transforming the k-th layer elasticity characteristics into a global system of the coordinates; z_k and z_{k-1} – the coordinates of the k-th layer.

The finite element is defined as a combination of bending (DKT) and membrane (CST) finite elements [10]. The DKT element [20] has 3 nodes and 3 degrees of freedom, ie a bending flexure and 2 rotations per node, the interpolation functions of the element should meet only C° continuity requirements, because only the first derivatives of the main variables – slopes to the middle surface – appear in the energy functional. The CST element has three nodes, the 1-st order interpolation function and two degrees of freedom – displacements per node.

Structural element matrices are generated based on the matrices of the membrane (*pl*) and bending (*b*) elements by combining them as required by the arrangement of the element degrees of freedom to produce a global element matrix.

The vector of degrees of freedom of any node of the element is expressed as:

$$\boldsymbol{\delta}_i = \{u_i, v_i, \omega_i, \theta_{xi}, \theta_{yi}, \theta_{zi}\}^T , \qquad (10)$$

where u_i, v_i, ω_i are node displacements; $\theta_{xi}, \theta_{yi}, \theta_{zi}$ – node rotations about the axis of the coordinates.

The stiffness matrix of the element is as follows:

$$[K_0] = \begin{bmatrix} [K_0^{pl}] & [K_0^{plb}] \\ [K_0^{plb}]^T & [K_0^{b}] \end{bmatrix} , \qquad (11)$$

where $[K_0^{pl}]$, $[K_0^{b}]$ and $[K_0^{plb}]$ are membrane, bending and coupling stiffness matrices.

$$[K_0^{pl}] = A[B_0^{pl}]^T[D^{pl}][B_0^{pl}] , \qquad (12)$$

where A is the area of the element; $[B_0^{pl}]$ – a linear membrane geometric matrix [10]. The elements of matrix $[K_0^{pl}]$ are calculated in the following way:

$$K_{0ij}^{pl} = \frac{1}{4A}\begin{bmatrix} d_{11}^{pl}b_ib_j + d_{33}^{pl}c_ic_j + & d_{12}^{pl}b_ic_j + d_{13}^{pl}b_ib_j + \\ d_{13}^{pl}(b_jc_i + b_ic_j) & d_{23}^{pl}c_ic_j + d_{33}^{pl}b_jc_i \\ d_{12}^{pl}b_jc_i + d_{13}^{pl}b_ib_j + & d_{22}^{pl}c_ic_j + d_{33}^{pl}b_ib_j + \\ d_{23}^{pl}c_ic_j + d_{33}^{pl}b_ic_j & d_{23}^{pl}(b_ic_j + b_jc_i) \end{bmatrix} ,$$

$$i = 1, 2, 3, \quad j = 1, 2, 3, \qquad (13)$$

where d_{ij}^{pl} denotes the elements of the accumulative membrane constitutive matrix $\left[D^{pl}\right]$; b_i and c_i are geometric coefficients. A global matrix is obtained from 9 sections of this type.

$$\left[K_0^b\right] = \int_V \left[B_0^b\right]^T \left[D^b\right]\left[B_0^b\right]dV , \qquad (14)$$

where $\left[B_0^b\right]$ is a linear bending geometric matrix [10].

After the rearrangement [18], any element of matrix $\left[K_0^b\right]$ is expressed in the following way:

$$K_{0ij}^b = \frac{1}{4A^2} \sum_{ii=1}^3 \sum_{jj=1}^3 \sum_{iii=1}^3 \sum_{jjj=1}^3 \Big[d_{11}^b X_{i,ii} X_{j,jj} b_{iii} b_{jjj} +$$

$$d_{22}^b Y_{i,ii} Y_{j,jj} c_{iii} c_{jjj} +$$

$$d_{12}^b \left(X_{i,ii} Y_{j,jj} b_{iii} c_{jjj} + X_{j,jj} Y_{i,ii} b_{jjj} c_{iii}\right)+$$

$$d_{13}^b \begin{pmatrix} X_{i,ii} X_{j,jj} b_{jjj} c_{iii} + X_{j,jj} Y_{i,ii} b_{iii} b_{jjj} + \\ X_{i,ii} X_{j,jj} b_{iii} c_{jjj} + X_{i,ii} Y_{j,jj} b_{iii} b_{jjj} \end{pmatrix}+$$

$$d_{23}^b \begin{pmatrix} X_{i,ii} Y_{j,jj} c_{iii} c_{jjj} + Y_{i,ii} Y_{j,jj} b_{iii} c_{jjj} + \\ X_{j,jj} Y_{i,ii} c_{iii} c_{jjj} + Y_{i,ii} Y_{j,jj} b_{jjj} c_{iii} \end{pmatrix}+$$

$$d_{33}^b \begin{pmatrix} X_{i,ii} X_{j,jj} c_{iii} c_{jjj} + X_{j,jj} Y_{i,ii} b_{iii} c_{jjj} + \\ X_{i,ii} Y_{j,jj} b_{jjj} c_{iii} + Y_{i,ii} Y_{j,jj} b_{iii} b_{jjj} \end{pmatrix}\Big] \times$$

$$\int \frac{\partial N_{iiz}}{\partial L_{iii}} \frac{\partial N_{jjz}}{\partial L_{jjj}} dA ,$$

$$i = 1, 2, …, 9, \quad j = 1, 2, …, 9, \qquad (15)$$

where d_{ij}^b denotes the elements of accumulative bending constitutive matrix $\left[D^b\right]$; N_i denotes the interpolation functions; $X_{i,ii}$ and $Y_{i,ii}$ are coefficients of the interpolation functions; b_{iii} and c_{iii} stand for geometric coefficients. The values of the integrals $\int_A \frac{\partial N_{iiz}}{L_{iii}} \frac{\partial N_{jjz}}{L_{jjj}} dA$, $\int_A N_{iiz} N_{jjz} N_{kkz} N_{llz} dA$, $\int_A \frac{\partial N_{jjz}}{L_{jjj}} dA$, $\int_A N_{iiz} N_{kkz} \frac{\partial N_{jjz}}{\partial L_{jjj}} dA$ are calculated by the software package Mathematica and stored in the data files. To retrieve them, indices $iiz = \frac{i-3}{3}3 + ii$ and $jjz = \frac{j-1}{3}3 + jj$, calculated according to integer calculation, are used. Then, the elements of the stiffness matrices are numerically synthesised. This method of matrix generation is more advantageous than numerical integration because it is time-saving and allows us to avoid the errors associated with numerical integration.

Flexural strains cause plane deformations, and vice versa (8). This effect is determined by a coupling stiffness matrix $\left[K_0^{plb}\right]$:

$$\left[K_0^{plb}\right] = \int_V \left[B_0^{pl}\right]^T \left[D^{plb}\right]\left[B_0^b\right]dV . \qquad (16)$$

When the above rearrangement is made, the elements of matrix $\left[K_0^{plb}\right]$ can be expressed as follows:

$$K_{0ij}^{plb} = \frac{1}{4A^2} \sum_{jj=1}^3 \sum_{jjj=1}^3 \Big[d_{11}^{plb} b_i X_{j,jj} b_{jjj} +$$

$$d_{12}^{plb} b_i Y_{j,jj} c_{jjj} + d_{23}^{plb} c_i Y_{j,jj} c_{jjj} +$$

$$d_{13}^{plb}\left(c_i X_{j,jj} b_{jjj} + b_i X_{j,jj} c_{jjj} + b_i Y_{j,jj} b_{jjj}\right)+$$

$$d_{33}^{plb}\left(c_i X_{j,jj} c_{jjj} + c_i Y_{j,jj} b_{jjj}\right)\Big]\times \int \frac{\partial N_{jjz}}{\partial L_{jjj}}dA ,$$

$$i = 1, 3, 5, \quad j = 1, 2, …, 9, \qquad (17)$$

$$K_{0\,(i+1)j}^{plb} = \frac{1}{4A^2} \sum_{jj=1}^3 \sum_{jjj=1}^3 \Big[d_{12}^{plb} c_i X_{j,jj} b_{jjj} +$$

$$d_{13}^{plb} b_i X_{j,jj} b_{jjj} + d_{22}^{plb} c_i Y_{j,jj} c_{jjj} +$$

$$d_{33}^{plb}\left(b_i X_{j,jj} c_{jjj} + b_i Y_{j,jj} b_{jjj}\right)+$$

$$d_{23}^{plb}\left(c_i X_{j,jj} c_{jjj} + c_i Y_{j,jj} b_{jjj} + b_i Y_{j,jj} c_{jjj}\right)\Big]\times \int \frac{\partial N_{jjz}}{\partial L_{jjj}}dA ,$$

$$i = 1, 3, 5, \quad j = 1, 2, …, 9, \qquad (18)$$

where d_{ij}^{plb} denotes the elements of the accumulative coupling constitutive matrix $\left[D^{plb}\right]$.

The matrix of the initial stresses of the element is as follows:

$$\left[K_\sigma\right] = \begin{bmatrix} 0 & 0 \\ 0 & \left[K_\sigma^b\right] \end{bmatrix}, \qquad (19)$$

$$\left[K_\sigma^b\right] = \int_V \left[G\right]^T \left[T\right]\left[G\right]dV , \qquad (20)$$

where matrix $\left[G\right]$ depends only on the coordinates [14]; $\left[T\right]$ is a matrix of the membrane stresses.

When the rearrangement is made [18], a single element of matrix $\left[K_\sigma^b\right]$ can be expressed as:

$$\left[K_{\sigma ij}^b\right] = \sum_{ii=1}^3 \sum_{jj=1}^3 \Big[X_{i,ii} X_{j,jj} T_x + Y_{i,ii} Y_{j,jj} T_y +$$

$$\left(X_{i,ii} Y_{j,jj} + Y_{i,ii} X_{j,jj}\right)T_{xy}\Big]\int_A N_{iiz} N_{jjz} dA , \qquad (21)$$

$$i = 1, 2, ..., 9, \quad j = 1, 2, ..., 9,$$

where T_x, T_y, T_{xy} are the membrane stresses.

The stiffness matrix of large displacements of the element is as follows:

$$\left[K_L\right] = \int_V \begin{bmatrix} 0 & \left[K_L^{plb}\right] \\ \left[K_L^{plb}\right]^T & \left[K_L^b\right] \end{bmatrix}dV , \qquad (22)$$

where $\left[K_L^b\right]$ and $\left[K_L^{plb}\right]$ are non-linear matrices of bending and coupling stiffness.

$$\left[K_L^b\right] = \int_V \left[\left[B_0^b\right]^T\left[D^{plb}\right]^T\left[B_L^b\right] + \left[B_L^b\right]^T\left[D^{plb}\right]\left[B_0^b\right] + \left[B_L^b\right]^T\left[D^{pl}\right]\left[B_L^b\right]\right]dV.\tag{23}$$

$$\left[K_L^b\right] = \left[K_{LB}^b\right]^T + \left[K_{LB}^b\right] + \left[K_{LC}^b\right].\tag{24}$$

After the rearrangement [18], the expressions for individual elements of matrices $\left[K_{LB}^b\right]$ and $\left[K_{LC}^b\right]$ are as follows:

$$K_{LB\ i,j}^b = \frac{1}{2A}\sum_{k=1}^{9}\sum_{ii=1}^{3}\sum_{jj=1}^{3}\sum_{kk=1}^{3}\sum_{jjj=1}^{3}(\boldsymbol{\delta}_k^b \times$$
$$\left[\begin{pmatrix}X_{i,ii}X_{k,kk}d_{11}^{plb} + Y_{i,ii}Y_{k,kk}d_{21}^{plb} + \\ \left(Y_{i,ii}X_{k,kk} + X_{i,ii}Y_{k,kk}\right)d_{31}^{plb}\end{pmatrix}X_{j,jj}b_{jjj} + \right.$$
$$\begin{pmatrix}X_{i,ii}X_{k,kk}d_{12}^{plb} + Y_{i,ii}Y_{k,kk}d_{22}^{plb} + \\ \left(Y_{i,ii}X_{k,kk} + X_{i,ii}Y_{k,kk}\right)d_{32}^{plb}\end{pmatrix}Y_{j,jj}c_{jjj} + \tag{25}$$
$$\begin{pmatrix}X_{i,ii}X_{k,kk}d_{13}^{plb} + Y_{i,ii}Y_{k,kk}d_{23}^{plb} + \\ \left(Y_{i,ii}X_{k,kk} + X_{i,ii}Y_{k,kk}\right)d_{33}^{plb}\end{pmatrix}\times$$
$$\left.\left(X_{j,jj}c_{jjj} + Y_{j,jj}b_{jjj}\right)\right]\times\int_A N_{iiz}N_{kkz}\frac{\partial N_{jjz}}{\partial L_{jjj}}dA),$$
$$i = 1, 2, \ldots, 9,\ \ j = 1, 2, \ldots, 9.$$

$$K_{LC\ i,j}^b = \sum_{k=1}^{9}\sum_{l=1}^{9}\sum_{ii=1}^{3}\sum_{jj=1}^{3}\sum_{kk=1}^{3}\sum_{ll=1}^{3}\left(\boldsymbol{\delta}_k^b\boldsymbol{\delta}_l^b\times\right.$$
$$\left[X_{k,kk}X_{l,ll}\begin{pmatrix}X_{i,ii}X_{j,jj}d_{11} + X_{i,ii}Y_{j,jj}d_{13} + \\ Y_{i,ii}X_{j,jj}d_{31} + Y_{i,ii}Y_{j,jj}d_{33}\end{pmatrix} + \right.$$
$$Y_{k,kk}Y_{l,ll}\begin{pmatrix}Y_{i,ii}Y_{j,jj}d_{22} + Y_{i,ii}X_{j,jj}d_{23} + \\ X_{i,ii}Y_{j,jj}d_{32} + X_{i,ii}X_{j,jj}d_{33}\end{pmatrix} + \tag{26}$$
$$X_{k,kk}Y_{l,ll}\begin{pmatrix}X_{i,ii}Y_{j,jj}d_{12} + Y_{i,ii}X_{j,jj}d_{21} + \\ X_{i,ii}X_{j,jj}d_{31} + Y_{i,ii}Y_{j,jj}d_{32} + \\ Y_{i,ii}Y_{j,jj}d_{23} + X_{i,ii}Y_{j,jj}d_{33} + \\ X_{i,ii}X_{j,jj}d_{13} + Y_{i,ii}X_{j,jj}d_{33}\end{pmatrix}\right]\times$$
$$\left.\int_A N_{iiz}N_{jjz}N_{kkz}N_{llz}dA\right),$$
$$i = 1, 2, \ldots, 9,\ \ j = 1, 2, \ldots, 9.$$

$$\left[K_L^{plb}\right] = \int_V\left[B_0^{pl}\right]^T\left[D^{pl}\right]\left[B_L^b\right]dV,\tag{27}$$

where $\left[B_L^b\right]$ is a non-linear geometric bending matrix [11].

After the rearrangement [18], the following expressions are obtained for the elements of matrix $\left[K_L^{plb}\right]$:

$$K_{L\ i,j}^{plb} = \frac{1}{2A}\sum_{k=1}^{9}\sum_{jj=1}^{3}\sum_{kk=1}^{3}\boldsymbol{\delta}_k^b\times\left(X_{j,jj}X_{k,kk}\times\right.$$
$$\left(b_ld_{11}^{pl} + c_ld_{31}^{pl}\right) + Y_{j,jj}Y_{k,kk}\left(b_ld_{12}^{pl} + c_ld_{32}^{pl}\right) + $$
$$\left(Y_{j,jj}X_{k,kk} + X_{j,jj}Y_{k,kk}\right)\left(b_ld_{13}^{pl} + c_ld_{33}^{pl}\right)\right)\times \tag{28}$$
$$\int_A N_{jjz}N_{kkz}dA,$$

$$K_{L\ (i+1),j}^{plb} = \frac{1}{2A}\sum_{k=1}^{9}\sum_{jj=1}^{3}\sum_{kk=1}^{3}\boldsymbol{\delta}_k^b\times\left(X_{j,jj}X_{k,kk}\times\right.$$
$$\left(b_ld_{31}^{pl} + c_ld_{21}^{pl}\right) + Y_{j,jj}Y_{k,kk}\left(b_ld_{32}^{pl} + c_ld_{22}^{pl}\right) + $$
$$\left(Y_{j,jj}X_{k,kk} + X_{j,jj}Y_{k,kk}\right)\left(b_ld_{33}^{pl} + c_ld_{23}^{pl}\right)\right)\times \tag{29}$$
$$\int_A N_{jjz}N_{kkz}dA,$$

$$i = 1, 3, 5,\ \ l = 1, 2, 3,\ \ j = 1, 2, \ldots, 9,$$

where $\boldsymbol{\delta}_k^b$ is the displacement or turn of the bending element node (10).

A connection error of external and internal forces (5) is calculated in the following way:

$$\boldsymbol{\psi}_i = \left\{\begin{array}{c}\frac{1}{2}\left[K_L^{plb}\right]\boldsymbol{\delta}_i^b \\ \left[K_L^{plb}\right]\boldsymbol{\delta}_i^{pl} + \frac{1}{2}\left[K_{LB}^b\right]^T\boldsymbol{\delta}_i^b + \\ \left[K_{LB}^b\right]\boldsymbol{\delta}_i^b + \frac{1}{2}\left[K_{LC}^b\right]\boldsymbol{\delta}_i^b\end{array}\right\},\tag{30}$$

where i is the iteration; matrices $\left[K_L^{plb}\right]$, $\left[K_{LB}^b\right]$ and $\left[K_{LC}^b\right]$ are generated according to the calculated displacements $\boldsymbol{\delta}_i^{pl}$ and $\boldsymbol{\delta}_i^b$ in iteration (i–1).

Stresses are calculated in this way:

$$\boldsymbol{\sigma} = \left\{\begin{array}{c}\boldsymbol{\sigma}_0^{pl} \\ \boldsymbol{\sigma}_0^b\end{array}\right\} + \left\{\begin{array}{c}\boldsymbol{\sigma}_L^{pl} \\ \boldsymbol{\sigma}_L^b\end{array}\right\},\tag{31}$$

where $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}_L$ are vectors of linear and non-linear stresses.

$$\boldsymbol{\sigma}_0^{pl} = \left[D^{pl}\right]\left[B_0^{pl}\right]\boldsymbol{\delta}^{pl} + \left[D^{plb}\right]\left[B_0^b\right]\boldsymbol{\delta}^b,\tag{32}$$

$$\boldsymbol{\sigma}_0^b = \left[D^{plb}\right]^T\left[B_0^{pl}\right]\boldsymbol{\delta}^{pl} + \left[D^b\right]\left[B_0^b\right]\boldsymbol{\delta}^b,\tag{33}$$

$$\boldsymbol{\sigma}_L^{pl} = \frac{1}{2}\left[D^{pl}\right]\left[B_L^b\right]\boldsymbol{\delta}^b,\tag{34}$$

$$\boldsymbol{\sigma}_L^b = \frac{1}{2}\left[D^{plb}\right]^T\left[B_L^b\right]\boldsymbol{\delta}^b.\tag{35}$$

4. Numerical examples

The quality of the finite element developed for solving geometrically linear and non-linear static and eigenvalue problems will be demonstrated by standard

tests of bending plates. Square symmetrical and non-symmetrical laminated plates which were fixed or hinged were tested by applying concentrated or distributed loads. The calculation results were compared as dimensionless values.
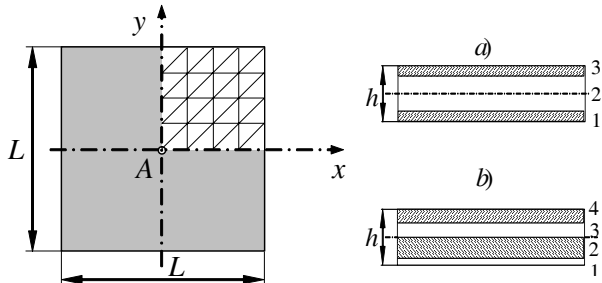

Fig 1. Computational schema: a) symmetrical laminated plate; b) non-symmetrical laminated plate

4.1. A linear problem

Test 1. A hinged square three-layer plate (Fig 1a) subjected to uniformly distributed pressure $q = 0,001$ was considered. The plate was calculated for 3 different materials making the intermediate layer. The difference between the material properties of the intermediate and upper and lower layers is 1 (further (1:1:1)), 10 (further (1:10:1)) and 50 (further (1:50:1)) times. The thickness of the layers is $h_1 = 10,0$, $h_2 = 80,0$ and $h_3 = 10,0$, respectively; L=1000,0. The orientation of the orthotropy axes of the layers is 0/45/0. The data on the properties of the material of the layers are given in Table 1.

Table 1. A symmetrical composite: properties of the materials of the layers

Layer No	Modulus of elasticity		Shear modulus	Poisson's ratio
	E_{11}	E_{22}	G_{12}	v_{12}, v_{21}
1, 3	3,4156	1,7931	1,0	0,44
a) laminated orthotropic material **(1:1:1)**				
2	3,4156	1,7931	1,0	0,44
b) properties of the material of the layer **(1:10:1)**				
2	0,34156	0,17931	0,1	0,44
c) properties of the material of the layer **(1:50:1)**				
2	0,06831	0,03586	0,02	0,44

One fourth of the plate was modelled by 72, 128 and 200 finite elements DKT_CST. A comparison of the calculation results obtained at the deflection point A (Fig 1) by using the DST element and analytical solutions [21] is provided in Table 2.

As shown in Table 2, the results obtained by using the element DKT_CST are monotonically approaching the accurate analytical and DST-based solutions. The accurate deflection values were obtained by subdividing a quarter of the plate into 128 elements.

Table 2. Central deflection of symmetric layered plate

Test	Solutions ω (according to the number of elements)				
	DST	Ana-lytical	DKT_CST		
	72		72	128	200
1:1:1	166,94	168,38	166,93	167,56	167,78
1:10:1	30,96	31,24	30,955	31,071	31,088
1:50:1	6,77	6,76	6,702	6,723	6,729

Test 2. Two laminated non-symmetrical square fixed and hinged plates were considered. The plates consisted of 4 layers (Fig 1b) of overall thickness $h = 0,04$ and layers thickness $h_1 = 0,011$, $h_2 = 0,09$, $h_3 = 0,1$, $h_4 = 0,1$, respectively, and L =1,0. The data on the properties of the material of the layers are presented in Table 3. The orientation of the orthotropy axes of the layers is 0/90/0/90. The plates were acted upon by a concentrated force $P = 100,0$ applied at the point A.

Table 3. A non-symmetrical composite: properties of the materials of the layers

Layer No	Modulus of elasticity		Shear modulus	Poisson's ratio
	E_{11}	E_{22}	G_{12}	v_{12}, v_{21}
1	$3,0 \times 10^8$	$3,0 \times 10^8$	$1,2 \times 10^8$	0,25
2, 3, 4	$3,0 \times 10^7$	$3,0 \times 10^7$	$1,2 \times 10^7$	0,25

A fourth of the plate was modelled by 8, 18, 32, 50, 72, 128 and 200 finite elements DKT_CST. A comparison of the calculation results obtained by using the TRIPLT element at the deflection point A is provided in Fig 2. The solutions based on the use of the element TRIPLT were obtained when the calculations of the fourth part of the plate divided into 18 elements had been made.
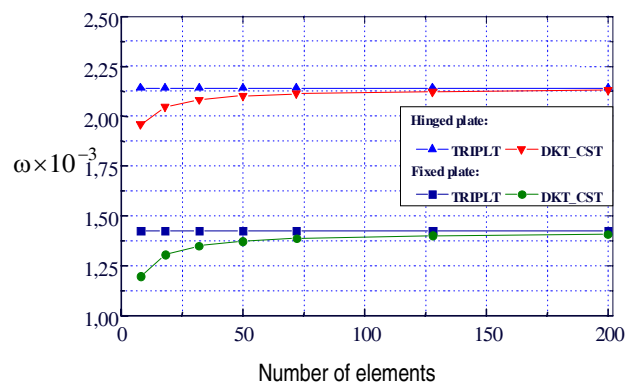

Fig 2. Convergence of values

One per cent difference between the solutions based on the use of TRIPLT and DKT_CST elements was obtained for a hinged plate by subdividing one-fourth of the plate into 128 DKT_CST elements. The results of similar accuracy were obtained for a fixed plate by subdividing one-fourth of the plate into 200 DKT_CST elements.

4.2. A geometrically non-linear problem

The calculations were made for a fixed non-symmetrical laminated square plate (Fig 1b). The plate consisted of 3 layers of the thickness: $h_1 = 0,1$, $h_2 = 0,65$, $h_3 = 0,25$; $L = 10,0$. The data on the properties of the material of the layers are presented in Table 4. The orientation of the orthotropy axes of the layers was 0/90/0. The deflection of the plate at point A was considered by changing the value of the concentrated load P from $0,01h$ to $0,7h$.

Table 4. Properties of the materials of the layers

Layer No	Modulus of elasticity		Shear modulus	Poisson's ratio
	E_{11}	E_{22}	G_{12}	ν_{12}, ν_{21}
1, 3	3,4156	1,7931	1,0	0,44
2	1,7931	3,4156	1,0	0,44

One-fourth of the plate was modelled using 128 DKT_CST finite elements. The deflections of the plate observed at point A were compared with the solutions obtained for the triangular element SHELL91 by calculating plates subdivided into 128 elements, using the software package ANSYS.

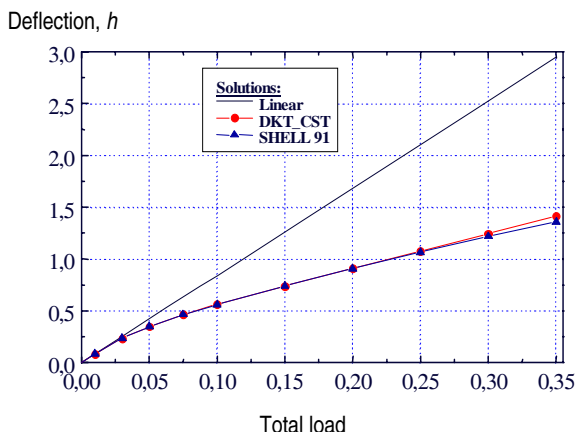The values of deflections are given in Fig 3 in terms of the overall plate thickness h.

Fig 3. Central deflection of the plate

The calculations show that the membrane strains occur when the deflection at point A is more than $0,2\ h$. When the linear deflection was $0,25\ h$, the difference between the non-linear solutions obtained by using the elements DKT_CST and SHELL91 made 0,08 %. Then, the linear deflection reached $2,5\ h$, while the difference between the non-linear solutions was 2 %.

4.3. The eigenvalue problem

The calculations were made for a non-fixed square three-layer plate (Fig 1). The whole plate was analysed to take into account all forms of oscillations, while the conditions of symmetry were neglected. The thickness of the plate layers was $h_1 = 0,1$, $h_2 = 0,15$, $h_3 = 0,25$, respec-

tively; $L = 4,0$; moduli of elasticity were $E_{11} = E_{22} = 2,0 \times 10^6$; Poisson's ratio was $\nu_{12} = \nu_{21} = 0,3$; shear moduli were $G_{12} = 0,77 \times 10^6$ and density $\rho = 1000$. The orientation of the layers orthotropy axes was 0/0/0.

The plate was modelled using 8, 32, 72, 128 and 200 DKT_CST elements. A comparison of the first three calculation results relating to non-zero eigen frequencies with analytical solutions is presented in Fig 4. The first six zero forms of the plate match the movements of a solid body. Non-zero forms are shown in Fig 5.
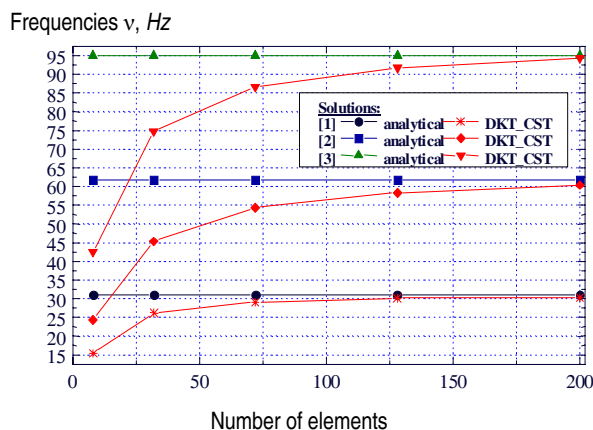

Fig 4. Convergence of the first three non-zero frequencies

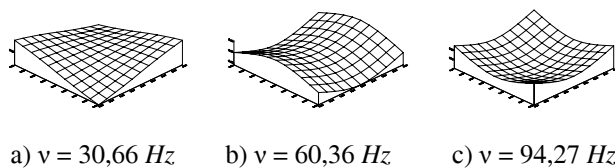a) $\nu = 30,66\ Hz$ b) $\nu = 60,36\ Hz$ c) $\nu = 94,27\ Hz$

Fig 5. Forms and frequencies of the first three non-zero natural oscillations

The convergence curves of the results presented in Fig 4 demonstrate that the solutions obtained using DKT_CST element quickly converge towards an accurate solution. The accurate value of the first eigen frequency was obtained by subdividing the plate into 128 elements.

5. Conclusions

All finite element structural matrices were generated using an effective analytical-numerical method of matrix development. This helped to avoid difficulties and errors involved in currently used numerical integration, to increase the accuracy of calculations and to obtain analytical expressions which can be easily implemented in software programmes for generating the element stiffness, initial stress and large displacement matrices and vectors of stresses.

The suggested finite element was numerically tested by comparing the results obtained in solving geometrically linear and non-linear static as well as eigenvalue problems with the well-known analytical solutions or the data obtained by applying the finite element method. The

analysis of the numerical tests shows that the element has good convergence characteristics as well as being sufficiently accurate and saving the time of calculation.

The suggested finite element can be used for modelling laminated anisotropic bending plates, their separate zones or beams when shear strains are insignificant.

The program developed for modelling laminated bending plates can be used in design offices and at industrial enterprises.

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ĮRANKIS SLUOKSNIUOTŲ LENKIAMŲ PLOKŠTELIŲ MODALINEI ANALIZEI

E. Michnevič

S a n t r a u k a

Efektyvaus diskretinės Kirchhofo teorijos trikampio baigtinio elemento DKT pagrindu suformuluotas naujas baigtinis elementas lenkiamoms daugiasluoksnėms plokštelėms modeliuoti. Plokštelės gali būti sudarytos iš kelių bet kokia tvarka išdėstytų sluoksnių, kurių medžiaga gali būti skirtinga bei ortotropinė. Naujas trikampis baigtinis elementas turi 6 laisvumo laipsnius kiekviename mazge: 3 linijinius poslinkius ir 3 posūkius apie koordinačių ašis. Elemento matematinis modelis apima visus deformacijų ir įtempių efektus tiek elemento plokštumoje, tiek statmena šiai plokštumai kryptimi, išskyrus šlytį. Elementas gali būti naudojamas sluoksniuotoms lenkiamoms plokštelėms arba sijoms, kurioms šlyties įtaka nežymi, skaičiuoti. Darbe pateikti skaitiniai pavyzdžiai, gauti rezultatai palyginti su žinomais analitiniais ir skaitiniais sprendiniais.

Reikšminiai žodžiai: laminatai, kompozitai, kompozitinės struktūros, sluoksniuotosios plokštelės, tiesinė analizė, netiesinė analizė, baigtinis elementas.

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